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AN ELEMENTARY INTRODUCTION TO ELLIPTIC FUNCTIONS BASED ON THE THEORY OF NUTATION

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Introduction. Elliptic functions are a classical branch of mathematics, and recur in many problems of physics and engineering—either directly, or as a skeleton of more complicated problems. Yet it cannot be denied that they are known nowhere as widely as they deserve. In many problems when the elliptic functions could advantageously be used, or at least inspected as a possibly useful first approximation, it is usually possible to find an even simpler (and not always adequate) skeleton of the theory in trigonometry.

Perhaps this is so because the elliptic functions are traditionally presented not as a refinement on trigonometry, but as a subject in the theory of functions of a complex variable; trigonometry, however, is never presented—and in fact, seldom reviewed—as such a subject. Indeed, there is no need to encumber a presentation of the circular functions to a beginner by a simultaneous presentation of the hyperbolic functions and of all relations between these two sets of functions. Nor would such a presentation be particularly instructive: for the poles of sin and cos are at an imaginary infinity (and hence cannot be circled), while those of tan, cot, sec and cosec lie simply on the real axis.

Moreover, the elliptic functions are traditionally presented not as solutions of differential equations, but as inversions of certain integrals. This is as though one introduced trigonometry by defining the arcsin as an "incomplete circular integral of the first kind," or worse, as though π were introduced as "twice the complete circular integral." The tradition evolved in the pre-computer era, when the numerical solution of differential equations was considered just another theoretical method, and not a very practical method at that, of specifying a new function. The solution was considered completed only when reduced to quadrature or series and interpolation; and it was therefore considered natural that much algebraic work had to be done before one resorted to computations. In contrast, now, when torrents of new functions are being generated on machines and the bottlenecks lie in the analytical work, the definition of a function by its differential equations is not only the more convenient, but appears to be a basic method.

A student naturally expects that the elliptic functions are a generalization of the circular functions, and are therefore more complicated; but he hopes that the new labor would be, in proportion, no more than, say, going from a circle to an ellipse, or from plane trigonometry to the spherical trigonometry. Instead, he finds an entirely new approach, and a new theory, which only retrospectively reduces to trigonometry—and a rather advanced trigonometry at that—as the poles of the elliptic analogues of sin and cos recede to imaginary infinity. The forward step (from circular to elliptic functions) requires turning this infinity into a large finite number. It would have been easier if this step involved an appearance of something small which was zero before; as a matter of fact, it is the latter which is provided—in various disguises—by his physical problems.

The historical necessity for economy in computations has left us other pedagogical hurdles. Thus, the differential equations which the new functions satisfy are usually not displayed prominently at the outset (so that it is only after much study that the student realizes that the generating equations had been served up as the differentiation formulas). Even so, often a desire to avoid (economically) the systems of equations, and to define each function by its single equation, results in an extensive use of transformations which are hardly essential. Thus, it is not easy for the student to recognize that equations such as

$$s'' = -(1+k^2)s + 2k^2s^3 \quad \text{or} \quad (s')^2 = (1-s^2)(1-k^2s^2) \tag{1}$$

are supposed to be the answer to the problem of the pendulum,

$$\ddot{x} + \sin x = 0, \tag{2}$$

through the relations

$$ks = \sin \tfrac{1}{2}x;\ k = \tfrac{1}{2}\dot{x}_0 \text{ or } 2/\dot{x}_0;\ x_0 = 0. \tag{3}$$

In this classical problem he further finds that even after the tables of s have been located, much drudgery remains in converting s to x; if a machine is to be resorted to, after all, it is much easier to put on it (2), rather than (3); certainly, various refinements of the problem are easier with (2) than with (1). In one important aspect his disappointment is even more legitimate. To him (2) is a complete statement of the physical system in question, expressed in some natural units which are independent of the initial conditions; and this statement should be augmented by a separate bit of information, the statement of the initial conditions. But with (1), he finds, the initial conditions in the equation are fixed, and a cognizance of the initial conditions of the physical problem requires working out the value of the modulus k; this is as though a change of the initial conditions changed the "system." In a sense, perhaps, this is precisely the essence of a nonlinear physical system; and the first equation of (1), in fact, shows that the nonlinearity of (2) can be taken care of by an introduction of a single nonlinear term, which is somewhat as if the expansion $\sin x = x - \frac{1}{6}x^3$ were exact. But, to the student, (1) seems to mix unnecessarily the statement of the system with the statement of the initial conditions. Thus the theory does not seem to show much for his trouble.

Of course, there are better reasons for the elliptic functions than helping our hypothetical student with his simple pendulum; there are many who would see in this pedagogical situation no more than a spur to further study, who would emphasize the logic of the historical development of the theory of elliptic functions, and who would point out that an approach on a higher level will in the long run be the more fruitful. But all these things will come to the student in good time. The path to arousing his interest in mathematics need not lie in keeping the solution of (2) difficult; for, all too often, the result is simply discouragement (if so simple a problem requires this much theory, he may well ask, what will the further developments require?). An insistence on the more powerful

methods of approach may equally well be likened by him to a prohibition of vaulting the wall of a labyrinth.

In fact, a certain amount of redundancy of effort (such as the exercises) is inavoidable in any study. While of all the disciplines the body of mathematics is most nearly as a "tree" of the circuit theory, our personal body of knowledge is certainly more as a network. Therefore, there seems to be room for a simplified and inductive, introductory, rearrangement of the theory of elliptic functions: one that would start from particularly simple physical problems involving these functions, which could gradually be expanded as needed to meet most of the existing theory, and which might have to be followed by a regular deductive course. Only a bare introduction of this sort is attempted here; and the only possibly novel idea is that a theoretical access to a computing machine might not be without some effect on the student's mathematical growth.

Of physical problems involving the elliptic functions a particularly neat one is that of the nutations of a free rigid body.

Physical background. Let us start with Euler's dynamical equations,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2, \tag{4}$$

wherein we know that, given the initial conditions $(\omega_{10}, \omega_{20}, \omega_{30})$, the functions $\omega_1, \omega_2, \omega_3$ of time t can readily be produced. The attractiveness of this problem lies in the fact that the principal three elliptic functions are no more than these three solutions, suitably scaled. Our object will be merely to systematize the possible extreme multiplicity of such functions.

We shall label the right-handed triad of the principal axes of inertia of our rigid body in the order of increasing moments of inertia:

$$I_1 \leqq I_2 \leqq I_3; \tag{5}$$

and must note that the physical significance of the I_1, I_2, I_3 imposes upon them a further restriction: they must be such as can form a triangle,*

$$I_1 \geqq I_3 - I_2. \tag{5'}$$

In order to mark at once the intimate relation of the elliptic functions to the trigonometry, the student should work out the important two degenerate cases, in which the elliptic functions degenerate into the circular functions. These are the cases of axial symmetry: the *spindle* ($I_1 < I_2 = I_3$, $\omega_1 =$ constant) and the *disc* ($I_1 = I_2 < I_3$, $\omega_3 =$ constant). The relation (5') limits the extreme case of the disc to the case of *thin disc* ($I_1 = I_2 = \frac{1}{2}I_3$). The extreme case of the spindle is the case of *thin rod* ($I_1 \to 0$), and the border-line case between a "fat spindle" and a "thick disc" is the *sphere* ($I_1 = I_2 = I_3$); these two extreme particularizations are so simple they are not interesting. Another particularization of the general case is

* This is so because $I_1 = K_2 + K_3$, etc., where $K_3 = \int_m x_3^2 dm$, etc. The relation (5') states simply that $K_2 + K_3 \geqq K_2 - K_3$.

a different generalization of the thin disc: it is the case of *flat body* ($I_1 \neq I_2$, $I_1+I_2=I_3$), and does not lead to the circular functions.

Alternatively, noting only that because of (5) only the second coefficient in parenthesis in (4) is positive—and that the other two are negative—the student may at once expect the axes 1 and 3 to possess some similar properties not shared by the axis of the intermediate moment of inertia, the axis 2.

A great deal of information about the functions $\omega_1, \omega_2, \omega_3$ can be got simply by inspecting the trajectories of the vector $\boldsymbol{\omega}$ in our body-fixed coordinate system 1, 2, 3; they are called polhodes ("pole paths," the "pole" being really the instantaneous vector of the angular velocity of the body*). While (4) gives the polhodes as parametric curves with t as the parameter, it is instructive to eliminate this fourth variable, t, from these equations. Multiplying (4) by $\omega_1, \omega_2, \omega_3$, adding and integrating, we have

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \text{const.} = 2T, \text{ say,} \tag{6}$$

viz., the conservation of the kinetic energy; while multiplying by $I_1\omega_1, I_2\omega_2, I_3\omega_3$, adding and integrating, we have

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{const.} = L^2, \text{ say,} \tag{7}$$

viz., the conservation of the magnitude of the angular momentum.

Strictly, there are two more constants of motion; for the vector $\mathbf{L}$ of angular momentum of our body is conserved in direction (in space), as well as in magnitude. But this does not show in our body-fixed system 1, 2, 3. Rather, equations (4) state that the 1, 2, 3 system is at all times so oriented that $\mathbf{L}$, given in that system by the variable components $L_1=I_1\omega_1$, $L_2=I_2\omega_2$, $L_3=I_3\omega_3$, remains constant in the space-fixed, newtonian, coordinate system.

We might also note that (7) is somewhat more general than (6); for (7) asserts only an absence of exterior forces on our body, while (6) asserts the absence of both exterior and interior work-producing forces.

Both (6) and (7) are equations of ellipsoids for the vector $\boldsymbol{\omega}$. Rewriting them as

$$\left(\frac{\omega_1}{\sqrt{(2T/I_1)}}\right)^2 + \left(\frac{\omega_2}{\sqrt{(2T/I_2)}}\right)^2 + \left(\frac{\omega_3}{\sqrt{(2T/I_3)}}\right)^2 = 1,$$

$$\left(\frac{\omega_1}{L/I_1}\right)^2 + \left(\frac{\omega_2}{L/I_2}\right)^2 + \left(\frac{\omega_3}{L/I_3}\right)^2 = 1,$$

we see that the "momentum ellipsoid" has semiaxes $L/I_1, L/I_2, L/I_3$; while the "kinetic energy ellipsoid" has semiaxes $\sqrt{(2T/I_1)}$, $\sqrt{(2T/I_2)}$, $\sqrt{(2T/I_3)}$, and

* Polhodes should be viewed not so much as curves, as conical surfaces with the apex at the center of mass of the body. The nutation of the body can be viewed as a slipless rolling of these polhodes on stationary curves (or conical surfaces) called herpolhodes ("creeping pole paths"). In our force-free case the latter curves happen to lie in a plane.

thus is more nearly a sphere.*

The trajectory of ω now can readily be visualized as the intersection of these two ellipsoids. For our purposes the most interesting observation is simply that these intersections are closed curves; it is thus strongly suggested that the motion is periodic. Figure 1 illustrates a momentum ellipsoid (7), with several trajectories resulting from the intersection of this ellipsoid with several kinetic-energy ellipsoids (6); that is, all of these trajectories have the same angular momentum but different energies. The larger energies (that is, the larger kinetic-energy ellipsoids) are associated with trajectories near axis 1; the lowest, with those about axis 3. The trajectories are divided into two groups, those looping axis 1, and those looping axis 3 (in a space-fixed coordinate system, of course, it is either the axis 1 or the axis 3, that nutates about the stationary **L**). A steady spin (constant ω) is possible about each one of the axis 1, 2, 3; but the spin about 1 and 3 is "stable" in a certain† sense, while that about 2 is "unstable" (if disturbed, the trajectory goes into a large loop either about 1 or about 3). In all these cases the two ellipsoids are tangent to each other at the axis of spin; but in the two stable cases one ellipsoid is wholly inside the other, except for the two points of tangency; while at the axis 2 the ellipsoids intersect, and the points of tangency are of a different type. The speed of ω along each trajectory, as given by (4), is obviously very low at the sharpened corners of the loops near axis 2; thus the body spends a long time spinning with one end of axis 2 near the stationary **L**, then rapidly "nutates" till it reverses its position, etc. The dividing trajectory which appears as passing through the axis 2, approaches this axis asymptotically, or leaves it in the analogous manner. The diametrally opposite

* In this connection we might make some remarks which are rather getting ahead of our story; but the reader might return to this footnote later, as an exercise. In certain problems of which our present one is a skeleton, it is in fact convenient to transform the variables in such a way that one of these two ellipsoids is a sphere. In particular, in Russian texts on mechanics it is the "more fundamental" ellipsoid, (7)—rather than the fatter one, (6)—that is so transformed: one then inspects the motion, in the 1, 2, 3 system, of the (really stationary) vector **L**. Alternatively, one could inspect the motion of a vector whose magnitude is $\sqrt{(2T)}$, and whose physical significance is more obscure. In either case Poinsot's delightful geometrical interpretation of the nutations of a free body (whereof the essence is that one inspects the motion of the vector ω) is so altered that it is practically lost. When (7) becomes a sphere, in particular, the other ellipsoid resembles the "moment of inertia ellipsoid," rather than the body itself: the relation between the two ellipsoids becomes, so to say, inverted. The sense of existence of a plane corresponding to the plane in which Poinsot's herpolhodes lie becomes easier: it becomes simply a plane tangent to the sphere. But the herpolhodes collapse into a point, and the rolling of this sphere on this plane is no longer slipless; to regain something like Poinsot's interpretation, it becomes necessary to slide the herpolhodes in that plane. However, in either case the polhodes can readily be produced when needed.

† This stability is taken in the Lyapunov's sense. It should be noted that this is one case where the Lagrangian criterion of stability (an equilibrium occurring at an energy extremum, and being stable or not as this extremum is a minimum or maximum) is not applicable. Here at an energy maximum (axis 1) there is stability, while the instability occurs (at axis 2) when there is no energy extremum. Yet, the two stable cases are not exactly alike; if the system is extended to include some dissipation of energy, axis 1 becomes unstable, while axis 3 becomes asymptotically stable.

loops correspond, of course, simply to a reversal of **L**.

Since all trajectories intersect the 1–3 plane, all of them will be included if we take the initial conditions in that plane, between axis 1 and 3:

$$\omega_{20} = 0. \tag{8}$$

We should then expect that ω_1 and ω_3 are even functions of time, while ω_2 is an odd function. Noting that all trajectories have symmetries about the planes 1–2, 2–3, and 3–1, we should also expect certain symmetries about the instants of time represented by an integer number of quarter-periods.

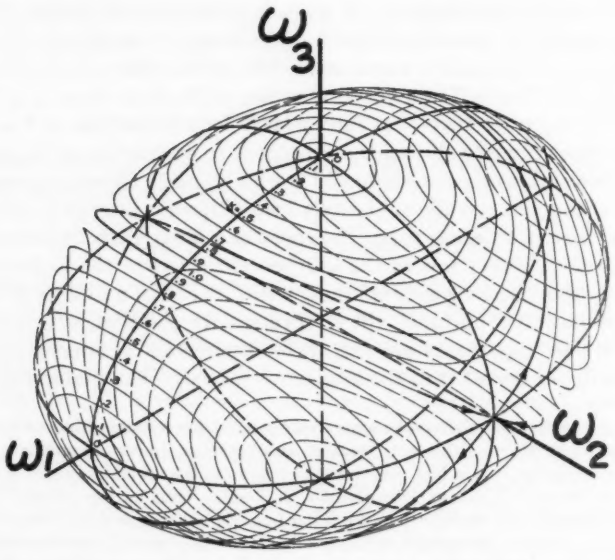


FIG. 1. Polhodes (the trajectories of the vector of angular velocity of the body in the body-fixed coordinates) of a free rigid body. In this case the ratios of the moments of inertia of the body are $I_1/I_2/I_3 = 3/4/5 (K_1/K_2/K_3 = 1/2/3)$. All polhodes shown have the same angular momentum.

We may note in passing that for these intersections to exist the L^2 and the $2T$ must be not entirely independent of each other: the maximum and the minimum semiaxes of the ellipsoids must be restricted by $\sqrt{(2T/I_1)} < L/I_1$ and $\sqrt{(2T/I_3)} > L/I_3$, otherwise written

$$L^2/I_1 > 2T > L^2/I_3 \quad \text{or} \quad 2TI_1 < L^2 < 2TI_3. \tag{9}$$

The two groups of trajectories which loop, respectively, axis 1 and 3 are correspondingly identified by $\sqrt{(2T/I_2)} > L/I_2$ and $\sqrt{(2T/I_2)} < L/I_2$ (*i.e.*, at the axis 2 the kinetic energy ellipsoid is either outside, or inside, the momentum ellipsoid). In the case

(10) $$\sqrt{(2T/I_2)} = L/I_2$$

(*i.e.*, the two ellipsoids are tangent to each other at the axis 2), one has both

$$2TI_2 = I_1I_2\omega_1^2 + I_2^2\omega_2^2 + I_3I_2\omega_3^2 = L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

and therefore

(11) $$I_1(I_2 - I_1)\omega_1^2 = I_3(I_3 - I_2)\omega_3^2$$

or

(12) $$|\omega_3/\omega_1| = \sqrt{\left(\frac{I_1}{I_3}\cdot\frac{I_2 - I_1}{I_3 - I_2}\right)} = \text{const.} = J, \text{ say,}$$

which shows, incidentally, that the trajectory dividing these two groups of loops does lie in a plane of slope J. The extremes of J occur at the two cases of axial symmetry and the situation at these extremes can be summarized by the following table:

Illustrated by	FIG. 2	FIG. 3
If I_2 approaches	I_1	I_3
the body becomes a	disc	spindle
whose axis is	3	1
Both ellipsoids become spheroids which are	oblate	prolate
The slope J of the dividing trajectory approaches	0	∞
so that this trajectory approaches the plane	1–2	2–3
and there vanish the loops about the axis	1	3
leaving only the circular loops about the axis	3	1
with the energy at these axes being a	minimum	maximum
Also, in the extreme cases (not illustrated) of a	thin disc	thin rod
when I_1 approaches	$\frac{1}{2}I_3$	0
the ratio of the spheroid diameter to its polar axis is	2	0

The solutions of (4) for the borderline case of (10) or (12) are further illustrated in Figure 4 (p. 602) and are described by (22) below. The functions ω_1 and ω_3 then are essentially sech t and ω_2 is tanh t.

Curiously, the requirements (9) have nothing to do with (5′). The mathe-

matical relations discussed—and those about to be discussed— hold even when (5′) is not satisfied and equations (4) cease to have their physical analog.

The intersection of the ellipsoids (6) and (7), forming one of the polhodes of Figure 1, is further illustrated in Figure 5 (p. 603).

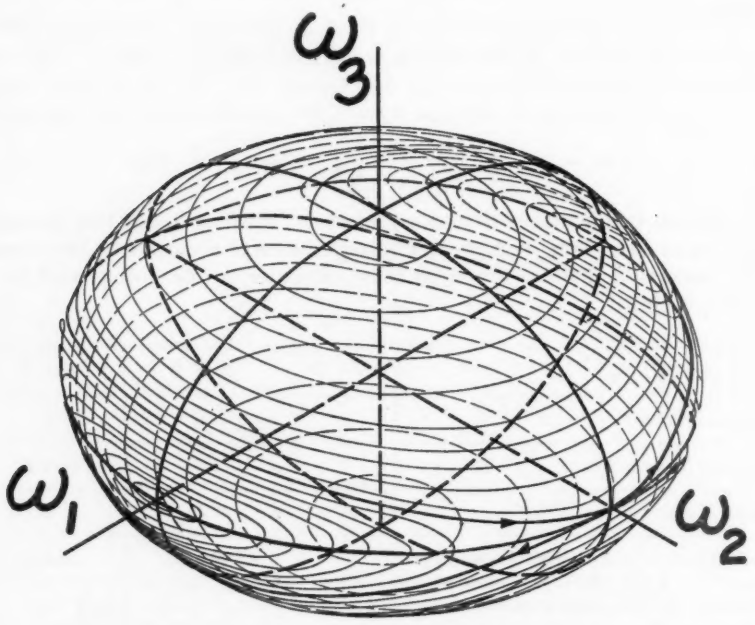


FIG. 2. Polhodes of a body which is nearly a disc ($I_1/I_2/I_3=3/3.2/5$). Note that the thickness (L/I_3) of the eventual oblate spheroid is at least one half of its diameter (L/I_2 or L/I_1) by (5′).

Generalization. The generalization of the solutions of (4) which results in the definition of elliptic functions amounts to no more than changing the scales of the variables ω_1, ω_2, ω_3, t; that is, to taking for these quantities some units other than radian/second and second. Let these quantities in the new units be W_1, W_2, W_3, u. The standard definition then is as follows.

It is natural to take for the even functions ω_1 and ω_3 the units of ω_{10} and ω_{30}, so that the maximum magnitude* of W_1 and W_3 is 1. For the unit of the odd function ω_2, let us take a quantity ω_{2M}, which we shall leave undefined for a while (presently it will turn out to be, usually, the maximum magnitude of ω_2). Finally, let us take the unit U of t in such a way that $\dot{\omega}_{20}$ in the new units is 1; and designate derivatives with respect to u by primes. Thus (Fig. 6, p. 605)

* That the minima and the maxima of (at least one of) W_1 and W_3 are equal in magnitude, follows from the symmetry of Figure 4. We avoid here the word "amplitude," for in the theory of elliptic functions it has a specialized meaning.

$$
\begin{aligned}
&W_1 = \omega_1/\omega_{10}, \qquad W_2 = \omega_2/\omega_{2M}, \qquad W_3 = \omega_3/\omega_{30}, \qquad u = t/U, \\
&\omega_1 = \omega_{10}W_1, \qquad \omega_2 = \omega_{2M}W_2, \qquad \omega_3 = \omega_{30}W_3, \qquad t = Uu,
\end{aligned} \tag{13}
$$

where $(') = d/du = Ud/dt = U(\cdot)$ $\quad (\cdot) = d/dt = (1/U)(')$. Equations (4) become

$$
\begin{aligned}
W_1' &= -\left\{U\frac{\omega_{2M}\omega_{30}}{\omega_{10}}\cdot\frac{I_3 - I_2}{I_1}\right\}W_2W_3, \\
W_2' &= +\left\{U\frac{\omega_{10}\omega_{30}}{\omega_{2M}}\cdot\frac{I_3 - I_1}{I_2}\right\}W_3W_1, \\
W_3' &= -\left\{U\frac{\omega_{10}\omega_{2M}}{\omega_{30}}\cdot\frac{I_2 - I_1}{I_3}\right\}W_1W_2.
\end{aligned} \tag{14}
$$

Now we can proceed in two ways, depending upon whether we are more interested in loops about axis 1 or 3.

1. Let us first favor the axis 3. The first two coefficients in the parenthesis

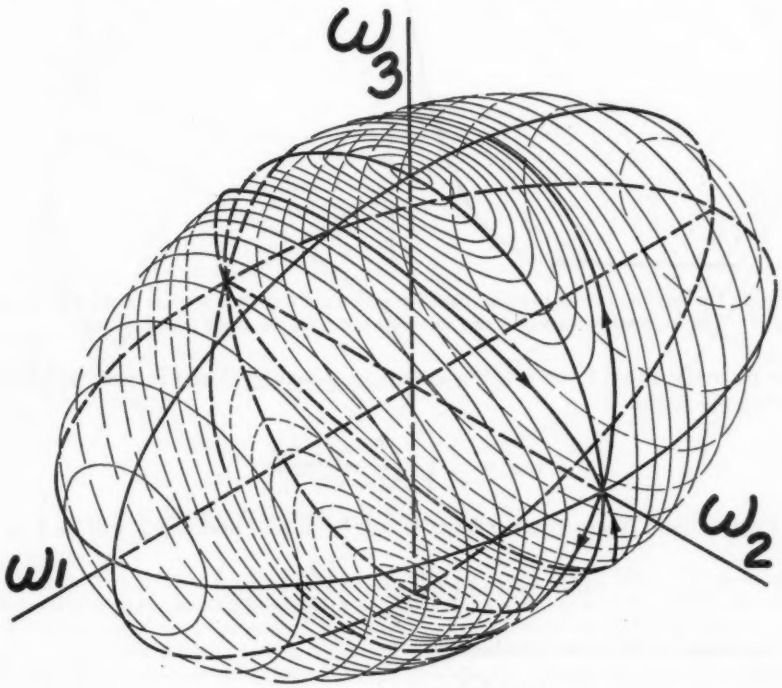


FIG. 3. Polhodes of a body which is nearly a spindle ($I_1/I_2/I_3 = 3/4.8/5$). Note that for a slender spindle ($I_1 \to 0$) the diameter (L/I_2 or L/I_3) of the eventual prolate spheroid can be negligible in comparison with its axis (L/I_1).

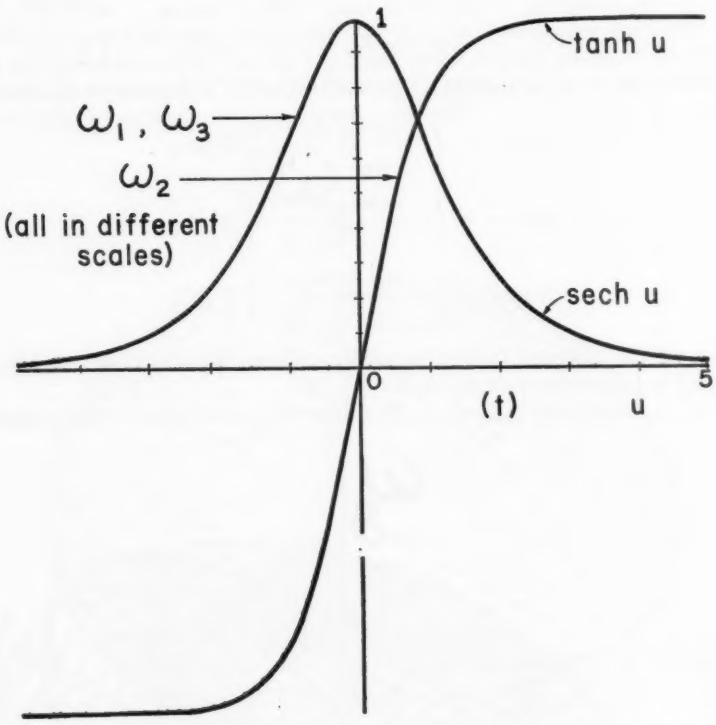


FIG. 4. The limiting case of unstable nutation shown starting from a point in the 1–3 plane. The trajectory approaches and/or comes out from the axis 2 asymptotically.

may be made equal if we select ω_{2M} in such a way that the ratio of these two coefficients is 1:

$$\omega_{2M} = \omega_{10}\sqrt{\left(\frac{I_3 - I_1}{I_3 - I_2}\cdot\frac{I_1}{I_2}\right)}. \tag{15}$$

Furthermore, these two coefficients will be 1, and hence W'_{20} will be 1, if

$$U = (1/\omega_{30})\sqrt{\left(\frac{I_1 I_2}{(I_3 - I_2)(I_3 - I_1)}\right)}. \tag{16}$$

The third coefficient then becomes

$$\left(\frac{\omega_{10}}{\omega_{30}}\right)^2 \cdot \frac{I_2 - I_1}{I_3 - I_2}\cdot\frac{I_1}{I_3} = k^2, \text{ say,} \tag{17}$$

and (4) acquire a relatively simple form

$$W_1' = -W_2W_3, \qquad W_2' = +W_3W_1, \qquad W_3' = -k^2W_1W_2; \tag{18}$$
$$W_{10} = 1, \qquad W_{20} = 0, \qquad (W_{20}' = 1), \qquad W_{30} = 1.$$

The number k which characterizes (18) thus reflects both a "system" (a combination of I_1, I_2, I_3) and the initial conditions (ω_{10}/ω_{30}, with $\omega_{20}=0$), and

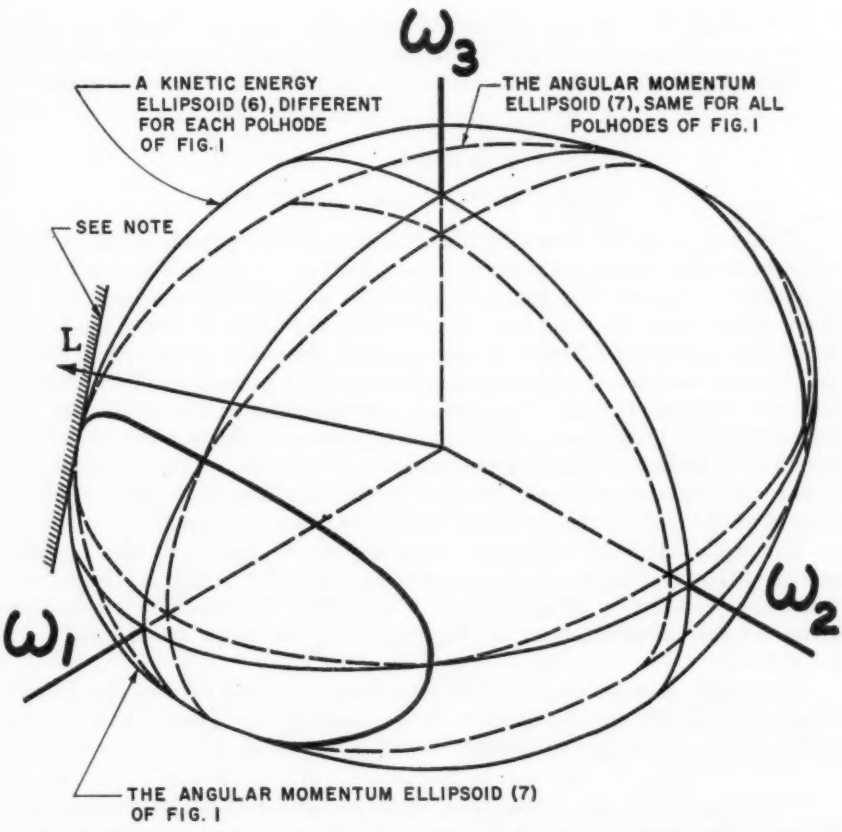

ω3
A KINETIC ENERGY ELLIPSOID (6), DIFFERENT FOR EACH POLHODE OF FIG. I
THE ANGULAR MOMENTUM ELLIPSOID (7), SAME FOR ALL POLHODES OF FIG. I
SEE NOTE
L
ω1
ω2
THE ANGULAR MOMENTUM ELLIPSOID (7) OF FIG. I

FIG. 5. Illustration of a pair of polhodes of Fig. 1 as an intersection of the pair of ellipsoids. (Note the herpolhode plane (a plane which is fixed, viz., not spinning with the coordinate system 1, 2, 3 of the body; normal to vector **L** of the angular momentum of the body; and tangent to the kinetic energy ellipsoid) for a particular polhode. This plane is drawn for a particular instant when it is seen edge on. The interesting property of the polhodes of a free rigid body is that, at all points on each polhode, the planes tangent to the kinetic energy ellipsoid are equidistant from the center of this ellipsoid. The motion of the body can therefore be visualized as a rolling of this ellipsoid, with its center stationary, on this plane. The herpolhodes encircle the point at which the vector **L** pierces this plane. Unlike the polhodes, the herpolhodes are in general not closed curves.

covers a large multiplicity of solutions of (4).

Now, the standard nomenclature is as follows:

$$
\begin{aligned}
&W_1 \text{ is called } \operatorname{cn} u && \text{(read: cosine-amplitude of } u\text{),*}\\
&W_2 \text{ is called } \operatorname{sn} u && \text{(read: sine-amplitude of } u\text{),*}\\
&W_3 \text{ is called } \operatorname{dn} u . && \text{(read: delta-amplitude of } u\text{).}
\end{aligned} \tag{19}
$$

k is called the modulus (the word "parameter" has a different specialized meaning in the theory of elliptic functions) and is usually reckoned as $0 \leqq k \leqq 1$.

Thus the elliptic functions are defined by the following differential equations

$$
\begin{aligned}
&(\operatorname{cn} u)' = -\operatorname{sn} u \operatorname{dn} u, && \operatorname{cn} 0 = 1;\\
&(\operatorname{sn} u)' = +\operatorname{cn} u \operatorname{dn} u, && \operatorname{sn} 0 = 0;\\
&(\operatorname{dn} u)' = -k^2 \operatorname{cn} u \operatorname{sn} u, && \operatorname{dn} 0 = 1.
\end{aligned} \tag{20}
$$

They are really functions of two variables, u *and* k; *e.g.*, cn u, which perhaps could be written as $\operatorname{cn}_k u$, is really $\operatorname{cn}(k, u)$. However, from the viewpoint of our hypothetical student who has some access to a computing machine, the dependence of these functions upon the modulus k is clinched only when he can write the generating differential equations for these functions with k as the independent variable. Such equations, however, are practically never written; nor are the elliptic functions functions of the complex variable $u+ik$. Yet, certain aspects of these functions which are closely related to their variability with k arise when one considers the functions of complex, or just imaginary, u. It is for this reason, perhaps, that the elliptic functions are traditionally considered to "belong" to the theory of functions of complex variable.

The coefficient k^2 is 0 at axis 3 (where $\omega_1=\omega_2=0$), and ∞ at axis 1 (where $\omega_3=\omega_2=0$). It is readily noted, by reference to (10) and (12), that for the polhode which divides the two groups of loops $k^2=1$. Thus the loops about axis 3 are given (in this interpretation) by $0 \leqq k^2 < 1$, and the loops about axis 1, by $1 < k^2 \leqq \infty$.

Clearly, the positive number k^2 is more convenient than the modulus k itself, which theoretically can be negative. If we consider the "great ellipse" in the plane 1–3 of Figure 1, the possible negative values of k may be associated

---

* Unfortunately, the more natural terms, "elliptic cosine" and "elliptic sine," together with the more natural notation, are apparently pre-empted by the characteristic functions ce_n and se_n of the Mathieu functions theory (which, however, has no analog of the third member of the triplet, the function dn u). In fact, the function ce_1 sometimes does resemble the platykurtic shape of sn, while se_1 sometimes resembles the leptokurtic shape of cn. This standard nomenclature is rather puzzling, since the Mathieu theory is both more complicated, and more recent. Basically, Mathieu's is a linear theory (even though the characteristic functions of that theory, superposable only with the secular, Ince's, modes, may be viewed as bordering on the nonlinear theory). The theory of elliptic functions, on the other hand, is distinctly nonlinear; it is only with the inversion of the problem that some resemblance to the superposability of solutions occurs, in that a certain integral is "expandable" in terms of several rational functions and the three elliptic integrals.

simply with the other half of either loop. As we proceed around this ellipse (or, as we proceed, at any point of our ellipsoid, at right angles to polhodes), the values of k obviously repeat periodically. But as it happens, the quantity which increases monotonically as we so proceed is the imaginary part of u.

2. Had we favored axis 1, we would have made the second two coefficients in (14) unity, obtaining, say

$$\tilde{\omega}_{2M} = \omega_{30}\sqrt{\left(\frac{I_3 - I_1}{I_2 - I_1}\cdot\frac{I_2}{I_2}\right)}, \tag{15'}$$

$$\overline{U} = (1/\omega_{10})\sqrt{\left(\frac{I_2 I_3}{(I_2 - I_1)(I_3 - I_1)}\right)}, \tag{16'}$$

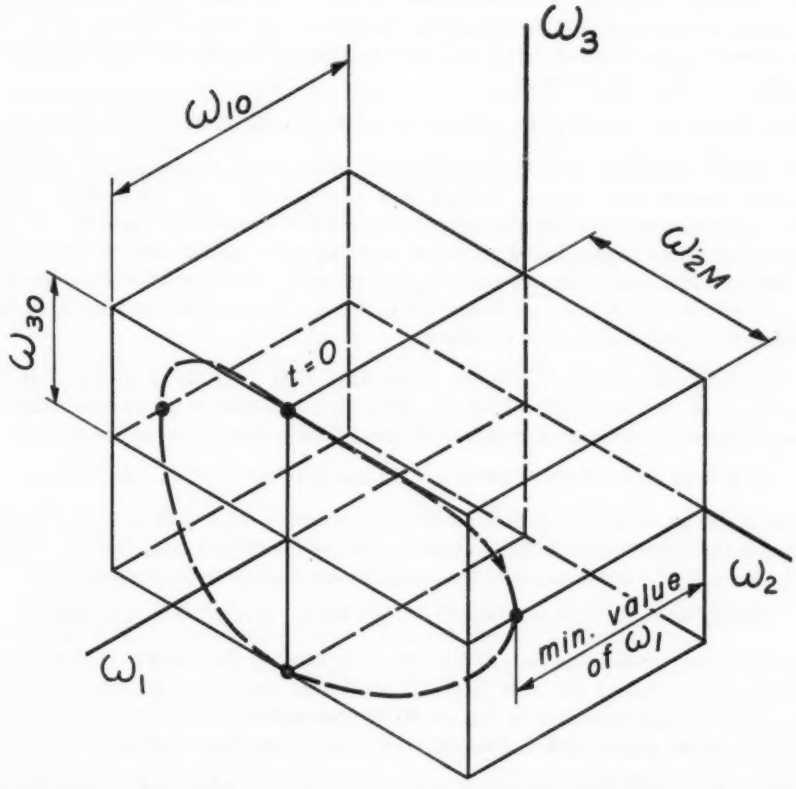


FIG. 6. Illustration of the units of ω_1, ω_2, ω_3, for a particular trajectory of Fig. 1 looping the axis 1. The unit of time is so chosen that the speed of the trajectory when crossing the axis 1 is unity. Note that for trajectories looping the axis 3, a different set of units is used.

and would have obtained another modulus,

$$\bar{k} = \frac{\omega_{30}}{\omega_{10}} \sqrt{\left(\frac{I_3 - I_2}{I_2 - I_1} \cdot \frac{I_3}{I_1}\right)}, \tag{17'}$$

which is readily noticed to be the reciprocal of k. Then the loops about axis 1 would be given by $0 \leqq \bar{k}^2 < 1$, and the loops about axis 3, by $1 < \bar{k}^2 \leqq \infty$. The equations would have been

$$d\overline{W}_1/d\bar{u} = -\bar{k}^2 \overline{W}_2 \overline{W}_3, \quad d\overline{W}_2/d\bar{u} = +\overline{W}_3 \overline{W}_1, \quad d\overline{W}_3/d\bar{u} = -\overline{W}_1 \overline{W}_2, \tag{18'}$$

so that with a modification of the identification (19), *i.e.*, with

$$\overline{W}_1 \text{ becoming } \operatorname{dn} \bar{u}, \quad \overline{W}_2 \text{ remaining } \operatorname{sn} \bar{u}, \quad \overline{W}_3 \text{ becoming } \operatorname{cn} \bar{u}, \tag{19'}$$

we would have arrived at the same equations as (20), except for the reciprocal meaning of the modulus. It should be noted, though, that $\overline{W}_2 \neq W_2$, $\bar{u} \neq u$; for the re-working of (18') into (18) requires putting $\bar{u}\bar{k} = u$, $\overline{W}_2/\bar{k} = W_2$, in addition to $\overline{W}_3 = W_1$, $\overline{W}_1 = W_3$, so that

$$\operatorname{cn}(1/k, u) = \operatorname{dn}(k, u/k), \quad \operatorname{sn}(1/k, u) = k \operatorname{sn}(k, u/k), \quad \operatorname{dn}(1/k, u) = \operatorname{cn}(k, u/k). \tag{19''}$$

The elliptic functions of modulus greater than 1, however, are usually not considered. Rather, one merely selects that one of (18) and (18') which gives $k^2 < 1$ (thereby making the first generalization for loops about axis 3, and the second for those about axis 1; that is, referring each loop to its own center).* It will be noted that with our numerical approach there is no objection (except for compliance with this convention) to $k > 1$. The transformation (13) for (19') is further illustrated in Figure 6.

Remarks on notation. While the generalization is a matter of choice of units, the choice of the "unitary" initial conditions is by no means imperative. Eagle† has proposed an alternative canonical form of these functions; in brief, it is

$$\operatorname{Sn} z = hk \operatorname{sn} u, \quad \operatorname{Cn} z = hk \operatorname{cn} u, \quad \operatorname{Dn} z = h \operatorname{dn} u, \quad u = hz, \quad h = K/\tau,$$

where K, as usual, is the quarter-period of sn u and cn u (or half-period of dn u), and τ is the long-needed abbreviation for $\frac{1}{2}\pi$ (to be pronounced "hi" for "half-pi"). Some of the advantages of his notation are that (20) becomes simply

$$(\operatorname{Cn} z)' = -\operatorname{Sn} z \operatorname{Dn} z, \quad (\operatorname{Sn} z)' = \operatorname{Cn} z \operatorname{Dn} z, \quad (\operatorname{Dn} z)' = -\operatorname{Cn} z \operatorname{Sn} z$$

and that many analogous simplifications occur later in the theory; that the real period of z is always 2π; and (for our purposes) that the variation of Sn z, etc., with k (and particularly for small k) does reflect, to a great extent, the variation of ω_1, ω_2, ω_3 with k. On the other hand, a student working with, say,

* Clearly, the same maximum value of ω_2, which is $\omega_{2M}(\operatorname{sn}_k u)_{\max} = \bar{\omega}_{2M}(\operatorname{sn}_{\bar{k}} \bar{u})_{\max}$, would be got from (18') as from (18); but this is not the same as to say that $\bar{\omega}_{2M} = \omega_{2M}$, nor that $(\operatorname{sn} u)_{\max}$ is always 1.

† Albert Eagle, The Elliptic Functions As They Should Be, Cambridge, 1958.

an analog computing machine might find the classical notation the more practical. Thus, the choice of "unitary" initial conditions may correspond happily to the limitation of machine voltage to + or −100 volts; while with such machine the re-scaling of the variables is always necessary, the variation of K with k might be of essence, and re-scaling from u to z, merely an extra labor; the blend with trigonometry (for $k \to 0$) in the standard notation may be the more nearly suggestive of the frequently used "linearization" of more complicated problems; and the fact that for the dividing polhode these functions approach infinity (with h) may seem an unnecessary further departure from practical applications. A great deal of other elegance of Eagle's notation would be appreciated by a more profound student of elliptic functions, rather than by their occasional user concerned only in recognizing these functions as a "skeleton" of his problem; in brief, by treating these functions as a self-contained—and internally highly consistent—subject, Eagle deepens, rather than resolves, the estrangement of the theory of elliptic functions from "practical" problems.

Basic properties of elliptic functions. When k of (17) approaches 0 (whether because the body approaches an axial symmetry, or because the vector ω approaches either one of axes 1 and 3), equations (20) with the same initial conditions reduce to those of circular functions:

$$\begin{aligned} W_1' &= -W_3W_2, & W_1 &= \operatorname{cn} u \to \cos u; \\ W_2' &= +W_3W_1, & W_2 &= \operatorname{sn} u \to \sin u; \\ W_3' &\to 0, & W_3 &= \operatorname{dn} u \to 1. \end{aligned} \tag{21}$$

When $k=1$ (*viz.*, for the dividing polhode) the dn u and the cn u merge, by (12) and (18′), and equations (20) reduce to hyperbolic functions, illustrated in Figure 4:

$$\begin{aligned} W_1' &\to -W_2W_1, & W_1 &= \operatorname{cn} u \to \operatorname{sech} u; \\ W_2' &= +W_1^2, & W_2 &= \operatorname{sn} u \to \tanh u; \\ W_3' &\to 0, & W_3 &= \operatorname{dn} u \to \operatorname{sech} u. \end{aligned} \tag{22}$$

In this sense the elliptic functions, as illustrated in Figure 7, may be viewed as a blend between the circular and the hyperbolic functions.

Multiplying the first two of equations (20) by cn u and sn u, adding, integrating and taking account of the initial conditions, we have

$$\operatorname{cn}^2 u + \operatorname{sn}^2 u = 1. \tag{23}$$

Hence the maximum magnitude of sn u is 1, and ω_{2M} of (15) or (15′)* is indeed the maximum magnitude of ω_2. In the classical notation, (23) is the only one in the theory of elliptic functions that is exactly the same as its analog in

* We are speaking here of the common case when $k<1$; *i.e.*, each loop on Figure 1 is considered with respect to its own center. For the "off-center" loops of the case $k>1$, the maximum value of sn u is $1/k$; formula (23) then, curiously, exchanges its physical meaning with (24).

trigonometry. All other analogs are more complicated.

Multiplying the second two of equations (20) by k^2 sn u and dn u, adding, integrating and taking account of the initial conditions, we have the definition of dn u (which was absent—or rather was simply 1—in trigonometry):

$$(24) \quad \mathrm{dn}^2 u = 1 - k^2 \mathrm{sn}^2 u = (1 - k^2) + k^2 \mathrm{cn}^2 u = \mathrm{cn}^2 u + (1 - k^2) \mathrm{sn}^2 u.$$

The quantity $k' = \sqrt{(1-k^2)}$ often recurring in this theory, is called the complementary modulus. Note that k' is not the same as $\bar{k}$ of (17′). Incidentally, with (23) and (24) the last equation of (1) can be recognized as the square of the second one of (20). Similar equations can easily be worked out for cn u and dn u.

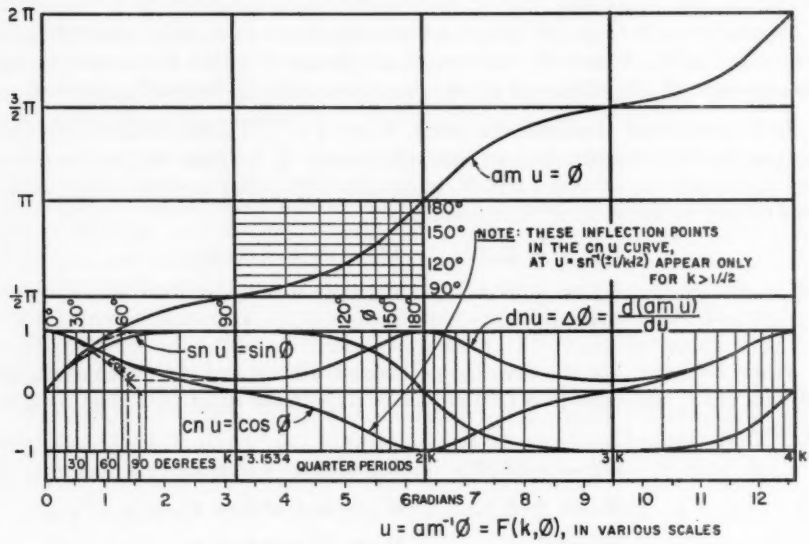


FIG. 7. Elliptic functions as stretched circular functions for $k = .9848 = \sin 80°$.

Because of (23) an angle, say ϕ, exists and is very important, such that cn $u = \cos \phi$ and sn $u = \sin \phi$. This angle is called the "amplitude" of u:

$$(25) \quad \phi = \mathrm{am}\, u = \cos^{-1}(\mathrm{cn}\, u), \qquad \mathrm{cn}\, u = \cos(\mathrm{am}\, u), \qquad \mathrm{sn}\, u = \sin(\mathrm{am}\, u).$$

The expression dn $u = \sqrt{(1-k^2 \mathrm{sn}^2 u)} = \sqrt{(1-k^2 \sin^2 \phi)}$ is often written $\Delta\phi$, and called "delta function" of ϕ, or delta function of amplitude (*viz.*, "delta-amplitude") of u. Obviously, the concept of such an angle is the main reason for the standard definition of elliptic functions.

Since the first two of equations (20) can be recognized as the formulas

$$d(\cos\phi)/du = -(\sin\phi)\cdot(d\phi/du), \qquad d(\sin\phi)/du = (\cos\phi)(d\phi/du),$$

it is obvious that

$$\text{dn } u = d\phi/du = d(\text{am } u)/du, \qquad \phi = \text{am } u = \int_0^u \text{dn } u \, du,$$

(26)

$$u = \int_0^{\text{am } u} \frac{d(\text{am } u)}{\text{dn } u} = \int_0^{\phi} \frac{d\phi}{\Delta\phi}.$$

Thus the elliptic functions are simply the functions cos ϕ and sin ϕ, whose abscissa ϕ is stretched nonuniformly into the function $u(\phi)$ such that the rate $d\phi/du$ is $\Delta\phi$ called dn u. At zeros of sine and extrema of cosine, the rate of u is the same as that of ϕ (*viz.*, dn $u = 1$); at extrema of sine and zeros of cosine, the rate of u is the maximum, and $d\phi/du$ has the minimum value, which is readily seen to be the complementary modulus k'. This is illustrated in Figure 7.

The value of u viewed as a function of am u and denoted by $F(k, \phi)$ is called the incomplete elliptic integral of the first kind. The term "integral" refers, of course, to the last expression of (26), and strongly connotes the pre-computer-era origin of this theory. The inverted function $u(\phi)$ is mathematically much more formidable than the more-relevant-to-all-sorts-of-physical-problems function $\phi(u)$: it is restricted to a certain interval of abscissa ϕ, is multivalued, and has in many places an infinite, or just very large, derivative. It seems amusing that precisely those classical methods (of inversion) which have been evolved to circumvent the labor of solving the differential equations are the ones which lead to difficulties with the computing machines. Thus in putting on the machines some practical problems whose skeleton involves the elliptic functions it may be well to think twice before proceeding with the refinement of these classical methods of inversion.

Commonly listed in the tables are the values of u for am $u = \frac{1}{2}\pi$, or the quarter-period of the motion, viewed as a function of an angle $\theta = \sin^{-1} k$ and called the complete elliptic integral of the first kind, $K(\theta)$. It varies from $\frac{1}{2}\pi$ for $\theta = k = 0$, to ∞ for $k = 1$, $\theta = \frac{1}{2}\pi$.

With their three reciprocals and the six ratios between them, the functions cn, sn, dn constitute the set of twelve basic, or Jacobian, elliptic functions. While six of them (such as tn $u = \tan \phi = \text{sn } u/\text{cn } u$ and ns $u = 1/\text{sn } u$) have obvious trigonometric analogs, the six involving dn (such as cd $u = \text{cn } u/\text{dn } u$) do not. There are many other elliptic functions; most of them, though, arose apparently either in sophisticated methods of computing the Jacobian functions or elliptic integrals, or as steps in the generalizations of the theory of doubly-periodic functions of complex variable. They are beyond the scope of this introduction.

Geometrical interpretation. The student may yet desire to have a simple geometric interpretation of elliptic functions, analogous to the classical trigonometric construction. It is perhaps the main pedagogical reason for the estrangement of the elliptic functions from trigonometry that no such construction

apparently exists. It may, perhaps, be sketched in principle as follows.

Let the unit circle AB on Figure 8a be shrunk vertically in the ratio $OC/OB = k' < 1$. The radius vector OD at an angle ϕ is shrunk in the ratio

$$\sqrt{\{\cos^2\phi + (k' \sin\phi)^2\}} = \sqrt{\{1 - (1 - k'^2)\sin^2\phi\}} = \sqrt{(1 - k^2\sin^2\phi)} = \Delta\phi,$$

so that the quantities $\operatorname{sn} u = \sin\phi$, $\operatorname{cn} u = \cos\phi$ and $\operatorname{dn} u = \Delta\phi$ can be recognized in the ratios FE/OC, OF/OA, OE/OA. Similarly, all other Jacobian elliptic functions can be recognized: thus, $\operatorname{cd} u = (OF/OA)/(OE/OA) = OF/OE$, etc. Given a value of k, we can compute the complementary modulus k', and construct the ellipse; and for each point E of this ellipse we can construct all Jacobian elliptic functions. However, they are still functions of the eccentric angle ϕ of the ellipse—that is, basically the trigonometric functions of the amplitude of u—rather than functions of u directly.

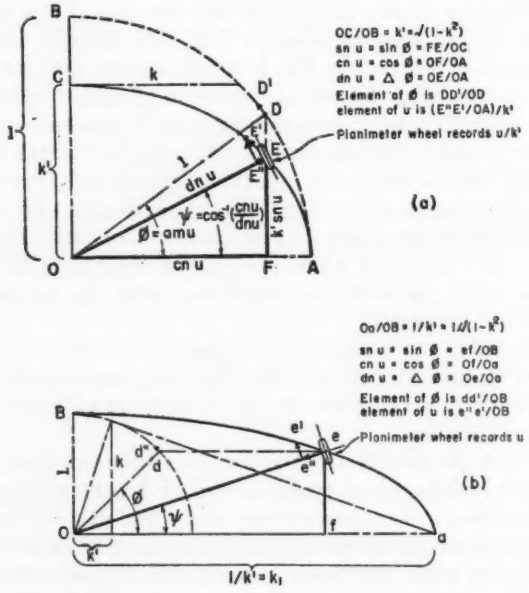

OC/OB = k' = √(1-k²)
sn u = sin ø = FE/OC
cn u = cos ø = OF/OA
dn u = Δ ø = OE/OA
Element of ø is DD'/OD
element of u is (E"E'/OA)/k'
Planimeter wheel records u/k'
(a)
Oa/OB = 1/k' = 1/√(1-k²)
sn u = sin ø = ef/OB
cn u = cos ø = Of/Oa
dn u = Δ ø = Oe/Oa
Element of ø is dd'/OB
element of u is e"e'/OB
Planimeter wheel records u
(b)

FIG. 8. Geometric interpretation of elliptic functions.

What, then, is—in this ellipse—the independent variable u? It is neither the arc AE (*that* can be expressed in terms of the more-commonly-met elliptic integral of the second kind), nor the angle AOE (*that* is a still different integral), nor the area of the elliptical sector (*that* is simply proportional to ϕ, or is a "circular integral").

This variable is the elliptic integral of the first kind,

$$u = F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\Delta\phi} = \int_0^{y} \frac{dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}},$$

where $y = \sin\phi$, which apparently is only met in the inversion of the problems stated as differential equations. While u cannot be readily recognized on Figure 8a, (nor can it be constructed directly with a ruler and a compass), its element—as shown by the integrand—can readily be constructed. It turns out that u is simply the circumferential travel of the tip of the radius vector, divided by k'. The travel DD' of D is always purely circumferential; but the displacement EE' of E consists of a radial displacement EE'' and a circumferential one $E''E'$. We may imagine a planimeter wheel, with its axis along the radius vector, which if attached to the radius vector at D would record the angle ϕ; if this planimeter now is constrained to slide upon the radius vector OE and be always located at E on the ellipse, its reading divided by k' will give u. The radial displacement of E on Figure 8a, of course, is simply $1-\mathrm{dn}\ u$, which vanishes in trigonometry.

Indeed, if we let the angle AOE be ψ, differentiating $\sin\psi = (k'y)/\sqrt{(1-k^2y^2)}$ and noting that $\cos\psi = \sqrt{(1-y^2)}/\sqrt{(1-k^2y^2)}$, we get

$$d\psi = \frac{k'dy}{(1-k^2y^2)\sqrt{(1-y^2)}}.$$

Clearly, the element of u is $1/k'$ times $\sqrt{(1-k^2y^2)}d\psi$ or $(OE)d\psi$. But this is the "circumferential travel" of E.

The factor $1/k'$ can be avoided by the equivalent construction shown on Figure 8b, in which the possible infinite value for the quarter-period K of u (for $k=1$) appears more naturally.

Further properties of the elliptic functions—addition formulas. Even in this sort of introduction it is necessary to mention the addition formulas, *i.e.*, to evaluate the functions of $u+v$ in terms of the functions of u and v. While they are not often needed, these formulas would serve for checking the oddness and evenness of these functions, and their reflectivity about the instants of u represented by the integer number of quarter-periods K; for deriving the formulas for the functions of $u-v$, $2u$, $\frac{1}{2}u$, etc.; and for "connecting" our simplified theory with that of the functions of complex variable. All of these formulas do possess some similarity to their trigonometric analogs, but are much more complicated. The most essential addition formulas are:

$$\begin{aligned} \mathrm{cn}\,(u+v) &= (\mathrm{cn}\,u\,\mathrm{cn}\,v - \mathrm{sn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,v)/(1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v), \\ \mathrm{sn}\,(u+v) &= (\mathrm{sn}\,u\,\mathrm{dn}\,v\,\mathrm{cn}\,v + \mathrm{cn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,u)/(1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v). \end{aligned} \tag{27}$$

Since

$$\mathrm{cn}\,u + i\,\mathrm{sn}\,u = \cos\phi + i\sin\phi = e^{i\phi} = e^{i\,\mathrm{am}\,u}, \tag{28}$$

it would have been preferable to have a simple formula for $\mathrm{am}(u+v)$; for then (27) could be easily derived by separating

$$e^{i\,\mathrm{am}\,(u+v)} = \mathrm{cn}\,(u+v) + i\,\mathrm{sn}\,(u+v) \tag{28'}$$

into real and imaginary parts. Unfortunately, no simple formula giving $\mathrm{am}(u+v)$ in terms of $\mathrm{am}\,u$ and $\mathrm{am}\,v$ seems to exist; but there is one in terms of $\mathrm{cn}\,u$, $\mathrm{cn}\,v$, $\mathrm{sn}\,u$, $\mathrm{sn}\,v$, $\mathrm{dn}\,u$, $\mathrm{dn}\,v$.

Formulas (27) may be derived on the basis of a very interesting analogy, which relates the elliptic functions to the spherical trigonometry. The amplitude of a sum $\mathrm{am}(u+v)$ is given by a side c of a spherical triangle whose other two sides are $a=\mathrm{am}\,u$ and $b=\mathrm{am}\,v$, provided that the angle C between these two sides is such that

$$\frac{\sin C}{\sin c} = k, \text{ and } C \text{ is obtuse.}^* \tag{29}$$

This analogy is based on the fact that if in the sum

$$u+v = \int_0^{\mathrm{am}\,u} d\phi/\Delta\phi + \int_0^{\mathrm{am}\,v} d\phi/\Delta\phi = \int_0^{\mathrm{am}\,(u+v)} d\phi/\Delta\phi$$

the addends u and v are varied without affecting the sum, the differentiation yields $d(\mathrm{am}\,u)/\mathrm{dn}\,u + d(\mathrm{am}\,v)/\mathrm{dn}\,v = 0$, which is analogous to the spherical trigonometric formula

$$\frac{da}{\cos A} + \frac{db}{\cos B} = 0 \tag{30}$$

for the spherical triangle whose sides a and b are varied in such a manner that the third side c and the angle C between a and b are kept as constants related by (29). That the ratio $\cos A/\cos B$ in that case is the same as $\mathrm{dn}\,u/\mathrm{dn}\,v$, follows from (29): for then $\sin A = k \sin a$ and $\sin B = k \sin b$, so that

$$\frac{\cos A}{\cos B} = \frac{\sqrt{(1-k^2\sin^2 a)}}{\sqrt{(1-k^2\sin^2 b)}} = \frac{\mathrm{dn}\,u}{\mathrm{dn}\,v}. \tag{29'}$$

Since C depends upon the unknown c, the derivation of (27) calls for some algebra, and the construction of such a spherical triangle cannot be undertaken directly. The possible variations of such triangle are illustrated, and certain details of the proof are spelled out, in Figure 9.

The convenient formula for the amplitude of a sum is

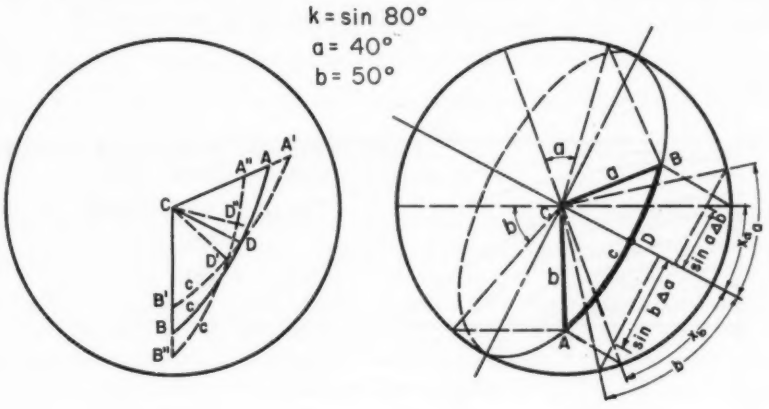
$$\mathrm{am}(u+v) = x_u + x_v, \tag{31}$$

where $x_u = \tan^{-1}(\mathrm{tn}\,u\,\mathrm{dn}\,v)$, $x_v = \tan^{-1}(\mathrm{tn}\,v\,\mathrm{dn}\,u)$ are the angles of the complex numbers

$$(\mathrm{cn}\,u + i\,\mathrm{sn}\,u\,\mathrm{dn}\,v) \quad \text{and} \quad (\mathrm{cn}\,v + i\,\mathrm{sn}\,v\,\mathrm{dn}\,u), \tag{32}$$

* With C acute, this procedure yields functions of $|u-v|$.

which are illustrated in Figure 9b and are certainly easy enough to remember: each one represents the radius vector OE of Figure 8a, reckoning the "shrinking factor" k' not as the complementary modulus, but as the dn of the other addend.



(a) (b)

FIG. 9. Spherical-trigonometric interpretation of (31).

(a) Variation of shape of triangle for c=const. (b) Relations in the view of the triangle (c) Derivation of (30): In $\cos c = \cos a \cos b - \sin a \sin b \cos C$, put c and C constant and differentiate. Thus:

$$(\sin a \cos b + \cos a \sin b \cos C)da + (\sin b \cos a + \cos b \sin a \cos C)db = 0.$$

Note that

$$\cos B = -(\sin a \cos b + \cos a \sin b \cos C)/\sin c,$$
$$\cos A = -(\sin b \cos a + \cos b \sin a \cos C)/\sin c,$$

from which (30) follows.

Formula (31) can be derived by noting that the right-hand side of (28′), formed from (27), can be factored as

$$\frac{(\operatorname{cn} u + i \operatorname{sn} u \operatorname{dn} v)(\operatorname{cn} v + i \operatorname{sn} v \operatorname{dn} u)}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{33}$$

the magnitude of which, of course, must be 1; indeed, the magnitude of each factor in the numerator is the square root of the denominator. *E.g.*,

$$\operatorname{cn}^2 u + \operatorname{sn}^2 u \operatorname{dn}^2 v = 1 - (1 - \operatorname{dn}^2 v) \operatorname{sn}^2 u = 1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v.$$

Thus (33) has the form

$$e^{i \operatorname{am}(u+v)} = (\cos x_u + i \sin x_u)(\cos x_v + i \sin x_v) = e^{i(x_u + x_v)}$$

which is (31).

All addition formulas, such as (27), can readily be derived from (31). A more direct *a priori* derivation of (31) would be of interest. It is also interesting to note that the arcs x_u and x_v, into which $c=\text{am}(u+v)$ is thereby broken, are the "spherical projections" of $a=\text{am}\ u$ and $b=\text{am}\ v$; that is, the point D in Figures 9a and 9b divides AB in such a way that the arc CD is normal to AB.

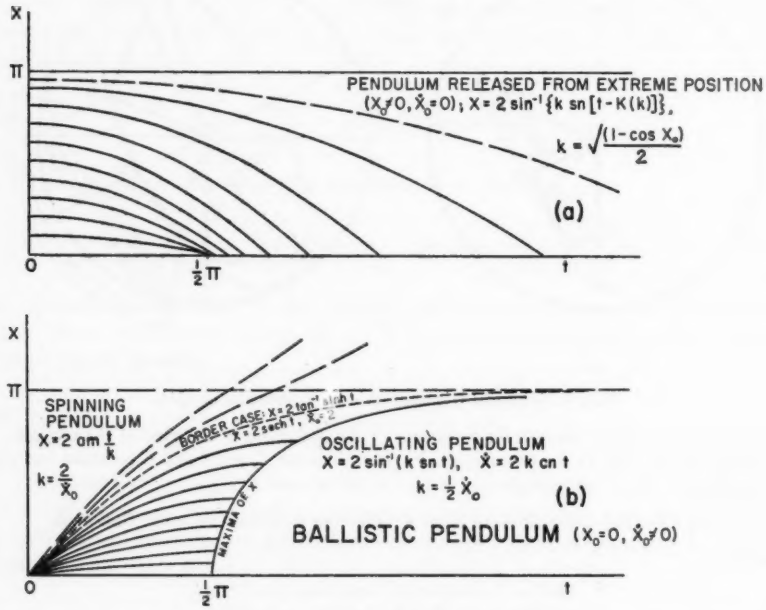


FIG. 10. A sample of a systematized set of solutions of (2). The remainder of the solution can be produced by the appropriate reflections. These "natural" curves are well known and are presented here merely as an example of the systematization of a practical problem treated with the aid of computing machines.

Remarks on the pendulum. Although the classical example (2) for the introduction of elliptic functions does not possess the symmetry of (4), it certainly describes the simpler physical phenomenon; for this reason, perhaps, this problem belongs to an introduction of this sort. It is particularly interesting for our purposes because it divides in two parts—oscillating and spinning pendulum—which correspond to the two groups of loops on Figure 1.

Faced with (2), our hypothetical student is apt to introduce some "elliptic functions" of his own: he would simply produce on a machine a systematized multiplicity of solutions $x(t)$, as in Figures 10a and 10b, and will be tempted to label these very "natural" functions the "elliptic cosine" and the "elliptic sine"; he would look into the theory of elliptic functions merely as a matter of con-

forming to the standard terminology. He will also note that the neater-looking plot of Figure 10a is limited (for obvious physical reason) to the oscillating pendulum, while the plot of Figure 10b can include also the case of the spinning pendulum; and would therefore concentrate on Figure 10b. After a heavy struggle with the theory (which would have to include, if only for lexicological reason, a nodding acquaintance with Mathieu functions) he will find that in the standard terminology his "natural" functions—of Figure 10b—have rather clumsy form:

(3′) $x = 2\sin^{-1}(k \operatorname{sn} t)$ for the oscillating pendulum,* with $k = \frac{1}{2}\dot{x}_0 < 1$;

(3″) $x = 2 \operatorname{am}(t/k)$ for the spinning pendulum, with $k = 2/\dot{x}_0 < 1$;

(22′) $x = 2\tan^{-1}(\sinh t)$, $\dot{x} = 2 \operatorname{sech} t$, with $\dot{x}_0 = 2$

for the borderline between these two cases (pendulum stopping asymptotically at the top). A far reaching socio-pedagogical reality is suggested here. Noting that his numerical approach (which merely reflects the physical reality) nicely blends the three cases which appear so distinct in the theory, our student is prone to develop a new appreciation of the machine, but to experience some corollary disenchantment toward the theory; the first is a healthy development, the second—in the long run—is certainly regrettable. In fact, (3′) and (3″) are equivalent, by (19″).

After all, the "theory" to him means no more than that "These sort of things have been done before, and are 'well known'; study the more general things, and you will be able to chop up these little problems quickly, without making a production of it." He would therefore be concerned with how, with a knowledge of (20) and (26), he could have recognized (2) as one leading to elliptic functions.†

Basically, he would now want to express the derivative in (2) as a product of two functions; it is therefore natural to put $x = 2y$ and write (2) as

$$\ddot{y} = -\sin y \cos y. \tag{2′}$$

The solution is easiest for the spinning pendulum (indeed, physically it is the simpler case, as nearer to a steady rotation; and technologically it is rather the more important case, as a skeleton of the operation of an engine). In this case we may simply let t be u on a certain scale, *e.g.*, let $t = Uu$ as in (13) and identify $\sin y$ and $\cos y$ with $\operatorname{sn} u$ and $\operatorname{cn} u$, y with $\operatorname{am} u$; in fact, we could write ϕ for y. By (26), y' is identified with $\operatorname{dn} u$, and hence y'' with $(\operatorname{dn} u)'$. Thus (2′) is at once identified with the third equation of (20), and there only remains to evaluate the constants. Since

$$y'' = U^2\ddot{y} = -U^2 \operatorname{sn} u \operatorname{cn} u = (\operatorname{dn} u)' = -k^2 \operatorname{sn} u \operatorname{cn} u,$$

* For Figure 10a, we have simply $x = 2\sin^{-1}(k \operatorname{sn}(t - K(k)))$; the concepts of cn *vs.* sn imply not only the change of phase, but also a change of the wave shape.

† The classical treatment (which proceeds by inverting the problem and introducing first the elliptic integral of the first kind), at least for the oscillating pendulum, is found in many texts, and need not be repeated here.

obviously, $U=k$. To evaluate it, we must consider the initial conditions:

$$x_0 = 0; \qquad \dot{x}_0 = (dx/dt)_0 = (dx/Udu)_0 = x_0'/U = 2y_0'/U = 2/U,$$

since $y_0' = \text{dn } 0 = 1$; hence $U=2/\dot{x}_0$. Thus $x=2y=2 \text{ am } u=2 \text{ am}(t/U)$; which is (3″).

With the oscillating pendulum the recognition is less easy (since the angle y is not monotonic, it cannot be identified with am u). Yet, it is intuitively obvious that the wave shape of sn u is relevant to the problem; we may expect it to represent x, y or sin y (note that it may not represent sin x, since $\ddot{x}$ must be small for x approaching π, while sn u is maximum). Let us try putting $\sin y = k \text{ sn } u$, where k is the maximum magnitude of sin y. Then

$$\cos y = \sqrt{(1 - (k \text{ sn } u)^2)} = \text{dn } u,$$

and (2′) becomes $\ddot{y} = -(k \text{ sn } u)(\text{dn } u)$; which begins to resemble the first equation of (20). The question then is whether $\ddot{y}/k$, which is y''/U^2k, can be identified with $(\text{cn } u)'$; or, whether an identification $y'/U^2k = \text{cn } u$ is consistent with our assumption $\sin y = k \text{ sn } u$. Differentiating the latter expression,

$$y' \cos y = k \text{ cn } u \text{ dn } u,$$

and cancelling $\cos y = \text{dn } u$, we have indeed $y' = k \text{ cn } u$; hence $U=1$. Our result, so far, is $\dot{x} = x' = 2y' = 2k \text{ cn } u$, with $k = \frac{1}{2}\dot{x}_0$ (since $\text{cn } 0 = 1$); this is the only simple result in this case. To find x, we must know how to integrate cn t; but for our present purposes the simplest way of doing that is to check the result by differentiating (3′).

THE INVARIANT FACTOR ALGORITHM

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R. V. Andree [1] gives a procedure for obtaining the inverse of a square matrix A. It amounts to reducing

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$$

by row and column operations to the form

$$\begin{pmatrix} P & I \\ 0 & Q \end{pmatrix}.$$

This implies that $PAQ=I$, whence $A^{-1}=QP$. In fact essentially the same procedure enables one to calculate the Smith canonical form of A when its entries

are drawn from a Euclidean ring (or even a principal ideal ring). Furthermore an analysis of the matrices P and Q which are by-products of this calculation provides one with an algorithm for many problems in module theory.

The invariant factor theorem states that if M is a free module over a principal ideal ring E, and if N is any submodule of M, then there exists a basis $x_1, \cdots, x_m$ of M, and scalars $e_1, \cdots, e_n$ in $E (n \leqq m)$ such that $e_1x_1, \cdots, e_nx_n$ form a basis of N, and e_i is a divisor of e_{i+1} for $i<n$. Furthermore the e_i, called invariant factors, are unique up to associates. The proof of this, together with a discussion of the applications, can be found in [2].

In proving the invariant factor theorem one begins with an arbitrary basis $x_1, \cdots, x_m$ of M. As N is a submodule of the free module M, N is free and so has a basis $y_1, \cdots, y_n$ (in practice the y's need only be a set of generators of N). Then

$$y_j = \sum_{i=1}^{m} a_{ij}x_i, \quad \text{where} \quad j = 1, \cdots, n, \quad \text{and} \quad a_{ij} \in E.$$

If the matrix (a_{ij}) is denoted by A, we can write this as $(y_1, \cdots, y_n) = (x_1, \cdots, x_m)A$. Observing that the operations (i) of multiplying x_i by a unit u of E, (ii) interchanging x_i and x_j, and (iii) adding any multiple of x_i to x_j have the effect of converting a basis into another basis, we apply these operations to the x- and y-bases and calculate the corresponding changes in A.

If $D(i, u)$ is the (unimodular) matrix obtained from the identity matrix I by multiplying u into the ith row of I, we have

$$(x_1, \cdots, x_i, \cdots, x_m) = (x_1, \cdots, ux_i, \cdots x_m)D(i, u^{-1}).$$

Thus multiplication of x_i by the unit u results in premultiplying A by the unimodular matrix $D(i, u^{-1})$. If the ith and jth rows of I are interchanged we get the unimodular matrix $R(i, j)$. Then

$$(\cdots, x_i, \cdots, x_j, \cdots) = (\cdots, x_j, \cdots, x_i, \cdots)R(i, j),$$

so that interchanging x_i and x_j amounts to premultiplying A by $R(i, j)$. Finally if $R(i+aj)$ denotes the matrix obtained from I by adding a times the jth row to the ith, we have

$$(\cdots, x_i, \cdots, x_j, \cdots) = (\cdots, x_i, \cdots, x_j + ax_i, \cdots)R(i - aj),$$

so adding ax_i to x_j amounts to premultiplying A by $R(i-aj)$. Note that if the matrices $D(i, u^{-1})$, $R(i, j)$, $R(i-aj)$ are premultiplied into A, they produce the corresponding row operations on A: $D(i, u^{-1})A$ has each element of the ith row of A multiplied by u^{-1}, $R(i, j)A$ has the ith and jth rows of A interchanged, $R(i-aj)A$ has a times the jth row of A subtracted from the ith. We also have

$$D(i, u^{-1})^{-1} = D(i, u), \quad R(i, j)^{-1} = R(i, j), \quad R(i - aj)^{-1} = R(i + aj).$$

If these matrices are postmultiplied into A, or the corresponding $n \times n$ ones

when A is an $m \times n$ matrix, we get the corresponding column operations performed on A in the cases of $D(i, u)$ and $R(i, j)$, but in the third case postmultiplication by $R(i+aj)$ has the effect of adding a times the ith column onto the jth.

Now it is proved that under the given hypotheses a succession of elementary row and column operations may be found to put A into diagonal form D with the invariant factors $e_1, \cdots, e_n$ down the main diagonal and all other entries zero. As each of these row (column) operations can be represented by pre(post)-multiplication by a unimodular matrix, and since the product of unimodular matrices is unimodular, we have the existence of unimodular matrices P and Q so that PAQ has the prescribed diagonal form D. To compute P and Q we write an $m \times m$ identity matrix I_m on the left of A and an $n \times n$ identity matrix I_n below:

$$\begin{pmatrix} I_m & A \\ & I_n \end{pmatrix}.$$

If a given row operation applied to A is also applied to I_m, it will convert I_m into a unimodular matrix R such that RA gives the result of the row operation on A. Similarly for column operations. If a sequence of row operations is carried out on A to produce B, then the same sequence will convert I_m into a unimodular matrix P such that $PA = B$. We have a similar statement for column operations. Applying both row and column operations to A to reduce it to D, we get the algorithm:

$$\begin{pmatrix} I_m & A \\ & I_n \end{pmatrix} \rightarrow \begin{pmatrix} P & D \\ & Q \end{pmatrix}.$$

A systematic procedure for carrying this out is to bring $e_1 = \text{g.c.d.}(a_{ij})$ to the top left position by elementary transformations, annihilate the remaining elements of the first row and column and proceed by recursion. In a Euclidean ring one has the Euclidean algorithm available to find the g.c.d. so that the above procedure can always be carried out in this case.

If y is a column matrix consisting of m scalars, then Py may be found by submitting y to the same row operations as A. We can carry it along with the main computation:

$$(*) \qquad \begin{matrix} I_m & I_m & A & y \\ & I_n & & \end{matrix} \rightarrow \begin{matrix} P^{-1} & P & D & Py. \\ & & Q & \end{matrix}$$

To find P^{-1} observe that if P is obtained by the elementary row operations with matrices $R_1, \cdots, R_r$ in that order, then $P = R_r R_{r-1} \cdots R_2 R_1$ so that $P^{-1} = R_1^{-1} \cdots R_r^{-1}$. It can be found by applying the inverse column operations to I_m while the row operations are being applied to A. Thus if the ith row of A is multiplied by u^{-1}, multiply the ith column of I_m (or its successor) by u; if a times the jth row of A is added to the ith, subtract a times the ith column of I_m

from the jth; if two rows of A are permuted, permute the corresponding columns of I_m, etc. This procedure is efficient but (it must be admitted) takes will-power.

In the following example E is taken to be the ring of integers, M the two-dimensional module of all ordered pairs of integers,

$$A = \begin{pmatrix} -18 & -22 \\ 12 & 14 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The form of the calculation is as in (*).

Set up the array:

$$\begin{array}{cccccc} 1\ 0 & 1\ 0 & -18 & -22 & 1 \\ 0\ 1 & 0\ 1 & 12 & 14 & -1. \end{array}$$

$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

Note that the g.c.d. of the entries of A is 2. We obtain 2 by applying the Euclidean algorithm to -22 and 14, the numbers in the second column of A. Hence we apply this algorithm in stages by means of row operations in the above array, using the inverse column operations on the leftmost matrix. To begin, add twice the second row to the first:

$$\begin{array}{cccccc} 1 & -2 & 1\ 2 & 6 & 6 & -1 \\ 0 & 1 & 0\ 1 & 12 & 14 & -1. \end{array}$$

$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

Then subtract twice the first row from the second:

$$\begin{array}{cccccc} -3 & -2 & 1 & 2 & 6\ 6 & -1 \\ 2 & 1 & -2 & -3 & 0\ 2 & 1. \end{array}$$

$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

Eliminate the 6 by subtracting three times the second row from the first:

$$\begin{array}{cccccc} -3 & -11 & 7 & 11 & 6\ 0 & -4 \\ 2 & 7 & -2 & -3 & 0\ 2 & 1. \end{array}$$

$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

Interchange rows and also columns:

$$\begin{matrix} -11 & -3 & -2 & -3 & 2 & 0 & 1 \\ 7 & 2 & 7 & 11 & 0 & 6 & -4. \end{matrix}$$

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

Thus

$$P = \begin{pmatrix} -2 & -3 \\ 7 & 11 \end{pmatrix}, \qquad Py = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the actual applications one omits the parts which are not needed.

As to the significance of P and Q, if $PAQ=D$, then reverting to the x- and y-bases,

$$(y_1, \cdots, y_n) = (x_1, \cdots, x_m)A = (x_1, \cdots, x_m)P^{-1}DQ^{-1},$$

whence

$$(y_1, \cdots, y_n)Q = (x_1, \cdots, x_m)P^{-1}D.$$

If we set

$$(y_1', \cdots, y_n') = (y_1, \cdots, y_n)Q, \quad (x_1', \cdots, x_m') = (x_1, \cdots, x_m)P^{-1},$$

then we have that

$$(y_1', \cdots, y_n') = (x_1', \cdots, x_m')D,$$

so that $y_k' = e_k x_k'$ for $k \leqq n$ as required by the invariant factor theorem. P^{-1} is thus the unimodular matrix giving the change in the x-basis and Q that giving the change in the y-basis.

The first application, to the solution of a linear Diophantine system, illustrates a broad class of commonly occurring problems. If $Ax=y$ is to be solved for x, where A is an $m \times n$, x an $n \times 1$, y an $m \times 1$ matrix of scalars belonging to an Euclidean ring E, put A into diagonal form by means of $PAQ=D$. Then $PAQQ^{-1}x=Py$, or $Dx'=y'$, where $x'=Q^{-1}x$ and $y'=Py$. If a solution of $Dx'=y'$ exists it is easily found. Having found x', we have $x=Qx'$. Observe that in this calculation we need Py rather than P. It may be set out as follows:

$$\begin{matrix} A & y \rightarrow & D & Py \rightarrow & Qx'. \\ I & & Q & x'. \end{matrix}$$

For example, to solve the system

$$3p + 2q - 4r = 14,$$
$$p + 4q - 2r = 28,$$

in integers. Set up the array:

$$\begin{matrix} 3 & 2 & -4 & 14 \\ 1 & 4 & -2 & 28 \\ 1 & 0 & 0 & . \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{matrix}$$

As the g.c.d. of the elements of A is 1, aim to isolate a 1 in a given row and column by making the remaining entries in that row and column zero. Here subtract three times the second row from the first:

$$\begin{matrix} 0 & -10 & 2 & -70 \\ 1 & 4 & -2 & 28 \\ 1 & 0 & 0 & . \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{matrix}$$

Subtract 4 times the first column from the second and add twice the first column to the third:

$$\begin{matrix} 0 & -10 & 2 & -70 \\ 1 & 0 & 0 & 28 \\ 1 & -4 & 2 & . \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{matrix}$$

Add five times the third column to the second:

$$\begin{matrix} 0 & 0 & 2 & -70 \\ 1 & 0 & 0 & 28 \\ 1 & 6 & 2 & . \\ 0 & 1 & 0 & \\ 0 & 5 & 1 & \end{matrix}$$

Interchange rows and columns:

$$\begin{matrix} 1 & 0 & 0 & 28 \\ 0 & 2 & 0 & -70 \\ 1 & 2 & 6 & . \\ 0 & 0 & 1 & \\ 0 & 1 & 5 & \end{matrix}$$

Fill in the solution, by inspection, in the lower right hand column:

$$\begin{array}{cccl} 1 & 0 & 0 & 28 \\ 0 & 2 & 0 & -70 \\ & & & \\ 1 & 2 & 6 & 28 = p' \\ 0 & 0 & 1 & -35 = q' \\ 0 & 1 & 5 & n = r' \ (n \text{ an arbitrary integer}). \end{array}$$

The solution

$$Qx' = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 28 \\ -35 \\ n \end{bmatrix} = \begin{bmatrix} -42 + 6n \\ n \\ -35 + 5n \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

A second application is to the analysis of the null space and range of a linear transformation on a module. Let M and N be free modules with bases respectively $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$. Let $T: M \to N$ be a linear transformation with matrix A. Thus $Tx_i = \sum_{k=1}^{n} a_{ki} y_k$, or

$$(y_1, \cdots, y_n)A = (Tx_1, \cdots, Tx_m) = T(x_1, \cdots, x_m).$$

If $PAQ = D$ then

$$(y_1, \cdots, y_n)P^{-1}D = (Tx_1, \cdots, Tx_m)Q = T(x_1, \cdots, x_m)Q.$$

Setting $(y_1', \cdots, y_n') = (y_1, \cdots, y_n)P^{-1}$ and $(x_1', \cdots, x_m') = (x_1, \cdots, x_m)Q$, we have new bases for N and M respectively such that the matrix corresponding to T is D. If $D = \operatorname{diag}(e_1, \cdots, e_r, 0, \cdots, 0)$ ($n \times m$, not square!) then $Tx_k' = e_k y_k'$ for $k \leqq r$ and $Tx_k' = 0$ for $r < k$. Hence the null space of T is spanned by $x_{r+1}', \cdots, x_m'$ and its range by $e_1 y_1', \cdots, e_r y_r'$. Referring elements of M and N to their coordinates with respect to the original x- and y-bases respectively, the null space is spanned by the last $n-r$ columns of Q and range by the first r columns of $P^{-1}D$. When E is a field and M and N are vector spaces, then $e_k = 1$ for $k \leqq r$ and the range is then spanned by the first r columns of P^{-1}.

A third application is the decomposition of a finitely generated module M into a direct sum of cyclic submodules. Let $g_1, \cdots, g_m$ be a set of generators for M and let $x_1, \cdots, x_m$ be a basis for a free module F over the same ring. There is a unique linear transformation $T: F \to M$ such that $Tx_i = g_i$. If N is the null space of T, a basis $x_1', \cdots, x_m'$ of F may be found such that $e_1 x_1', \cdots, e_n x_n'$ is a basis for N and e_i is a divisor of e_{i+1} for $i < n$. Putting $g_i' = Tx_i'$ the module M splits into a direct sum of cyclic submodules $(g_1') \oplus \cdots \oplus (g_m')$, where $(g_i') = E/e_i E$.

The computation starts with a set $y_1, \cdots, y_n$ of generators of N, the null space of T. A is defined by $(y_1, \cdots, y_n) = (x_1, \cdots, x_m)A$. As before, we have

$(y_1, \cdots, y_n)Q = (x_1, \cdots, x_m)P^{-1}D$ so that the required new basis of F is $(x_1', \cdots, x_m') = (x_1, \cdots, x_m)P^{-1}$. Thus the new generators required for the original module M are $(g_1', \cdots, g_m') = (g_1, \cdots, g_m)P^{-1}$. In the present application it is P^{-1} that we wish to find.

As a numerical example consider the abelian group G with generators g_1, g_2, g_3 and relations $53g_1 - 25g_2 - 9g_3 = 0$, $21g_1 - 10g_2 - 3g_3 = 0$, considered as a module over the integers. Let x_1, x_2, x_3 be a basis for the free module F. The null space of T is spanned by $y_1 = 53x_1 - 25x_2 - 9x_3$ and $y_2 = 21x_1 - 10x_2 - 3x_3$. Then

$$A = \begin{pmatrix} 53 & 21 \\ -25 & -10 \\ -9 & -3 \end{pmatrix}.$$

As we have to find P^{-1} only, write I to the left of A and as row operations are applied to A, apply the inverse column operations to I. Column operations may be freely applied to A without affecting the computation of P^{-1}. Thus

$$\begin{array}{ccc|cc} 1 & 0 & 0 & 53 & 21 \\ 0 & 1 & 0 & -25 & -10 \\ 0 & 0 & 1 & -9 & -3 \end{array} \rightarrow \begin{array}{ccc|cc} 1 & -2 & 0 & 3 & 1 \\ 0 & 1 & 0 & -25 & -10 \\ 0 & 0 & 1 & -9 & -3 \end{array}$$

on adding twice the second row to the first. We proceed as before to make the other entries in the row and column containing the 1 vanish, obtaining

$$\begin{array}{ccc|cc} 21 & -2 & 0 & 0 & 1 \\ -10 & 1 & 0 & 5 & 0. \\ -3 & 0 & 1 & 0 & 0 \end{array}$$

Interchanging the last two columns:

$$\begin{array}{ccc|cc} 21 & -2 & 0 & 1 & 0 \\ -10 & 1 & 0 & 0 & 5. \\ -3 & 0 & 1 & 0 & 0 \end{array}$$

Hence

$$P^{-1} = \begin{pmatrix} 21 & -2 & 0 \\ -10 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

so that $e_1 = 1$, $e_2 = 5$, $g_1' = 21g_1 - 10g_2 - 3g_3$, $g_2' = -2g_1 + g_2$, $g_3' = g_3$. So g_1' generates the zero subgroup, g_2' a cyclic subgroup of order 5, g_3' an infinite cyclic group. We have $G = (-2g_1 + g_2) \oplus (g_3)$.

The same technique permits the reduction of an $n \times n$ matrix A over a field

Φ to canonical form, enabling one to calculate a nonsingular matrix N such that $N^{-1}AN$ has that form. Here A acts as a linear operator over the vector space M of all n-rowed column matrices with entries in Φ. Taking the ring E to be $\Phi[t]$, the ring of all polynomials in an indeterminate t with coefficients in Φ, we make M into a module over E by setting $p(t)g = p(A)g$, where $p(t)$ is a polynomial in t and $g \in M$. This defines the product of a scalar $p(t)$ in E and a vector g of M. It is easily verified that with this definition of multiplication, M is a module over the ring E. Clearly if $g_1, \cdots, g_n$ is a basis for the *vector space* M (over Φ), it will also be a set of generators for the *module* M (over E). Let F be the free module with basis $x_1, \cdots, x_n$ over E; let T be the module linear transformation defined from F into M by $Tx_k = g_k$. Then the null space of T is spanned by

$$y_i = \sum_{j=1}^{n} (\delta_{ji}t - a_{ji})x_j, \qquad i \leqq n.$$

Thus we apply the invariant factor algorithm to the matrix $tI - A$ to obtain unimodular (polynomial) matrices $P(t)$ and $Q(t)$ such that $P(t)(tI - A)Q(t) = D(t)$, where the invariant factors of A appear as the diagonal elements of $D(t)$. The vectors $(g_1', \cdots, g_n') = (g_1, \cdots, g_n)P(t)^{-1}$ will generate the cyclic subspaces of M with respect to A, the different invariant factors $e_1(t), \cdots, e_n(t)$ being their respective annihilating polynomials.

Consider the example

$$A = \begin{pmatrix} -6 & 0 & 2 \\ -15 & 0 & 5 \\ -18 & 0 & 6 \end{pmatrix}.$$

To find $P(t)^{-1}$ set up the array

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & t+6 & 0 & -2 \\ 0 & 1 & 0 & 15 & t & -5 \\ 0 & 0 & 1 & 18 & 0 & t-6 \end{array}$$

where the entries are elements of the ring E, *i.e.*, polynomials in t. The g.c.d. of the right-hand matrix is 1, which must be the first invariant factor. Applying the Euclidean algorithm to the first two rows we get a 1 in the third column. Thus subtract twice the first row from the second and then multiply through the second row by -1 to get

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & t+6 & 0 & -2 \\ 2 & -1 & 0 & 2t-3 & -t & 1. \\ 0 & 0 & 1 & 18 & 0 & t-6 \end{array}$$

The remaining elements in the second row are now annihilated by column operations:

$$\begin{matrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 5t & -2t & -2 \\ 0 & 0 & 1. \\ -2t^2+15t & t^2-6t & t-6 \end{matrix}$$

Annihilating the remaining elements in the third column by row operations:

$$\begin{matrix} 1 & -2 & 0 \\ 2 & -5 & 0 \\ 0 & t-6 & 1 \end{matrix} \qquad \begin{matrix} 5t & -2t & 0 \\ 0 & 0 & 1. \\ -2t^2+15t & t^2-6t & 0 \end{matrix}$$

The g.c.d. of the remaining members is t. Add twice the second column to the first:

$$\begin{matrix} 1 & -2 & 0 \\ 2 & -5 & 0 \\ 0 & t-6 & 1 \end{matrix} \qquad \begin{matrix} t & -2t & 0 \\ 0 & 0 & 1. \\ 3t & t^2-6t & 0 \end{matrix}$$

Annihilate the remaining elements in the first row and column:

$$\begin{matrix} 1 & -2 & 0 \\ 2 & -5 & 0 \\ 3 & t-6 & 1 \end{matrix} \qquad \begin{matrix} t & 0 & 0 \\ 0 & 0 & 1. \\ 0 & t^2 & 0 \end{matrix}$$

Rearrange:

$$\begin{matrix} -2 & 1 & 0 \\ -5 & 2 & 0 \\ t-6 & 3 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & t & 0. \\ 0 & 0 & t^2 \end{matrix}$$

Thus

$$P(t)^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -5 & 2 & 0 \\ t-6 & 3 & 1 \end{bmatrix}.$$

Then $g_1' = 0$, $g_2' = g_1 + 2g_2 + 3g_3$ is annihilated by A and $g_3' = g_3$ is annihilated by A^2. If the set g_2', Ag_3', g_3' is taken as a basis for the vector space M, A will have the matrix representation

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which is the Jordan canonical form corresponding to the elementary divisors t and t^2. The analysis in the case where the invariant factors are products of

powers of distinct irreducible factors is more complex but runs on the same lines. As for N, its columns consist of the coordinates of the new basis of M with respect to the original basis. Thus

$$N = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 1 \end{pmatrix}.$$

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ARITHMETICAL FUNCTIONS OF GENERALIZED PRIMES

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Introduction. Generalized primes were first introduced by Beurling [1], and later developed by Nyman [2]. In this paper they will be defined as follows:—

Suppose given a finite or infinite sequence $\{p\}$ *of real numbers (generalized primes) such that*

$$1 < p_1 < p_2 < p_3 < \cdots.$$

Form the set $\{l\}$ *of all possible p-products, i.e., products* $p_1^{v_1}p_2^{v_2}, \cdots$, *where* v_1, $v_2, \cdots$ *are integers* $\geqq 0$ *of which all but a finite number are* 0.

Call these numbers "l numbers" and suppose that no two l numbers are equal (if their v's are different). Then arrange $\{l\}$ *as an increasing sequence:*

$$1 = l_1 < l_2 < l_3 < \cdots.$$

Examples.

(a) $\{p\}: 6 < 35 < 143 < 323 < \cdots$ (infinite).

$\{l\}: 1 < 6 < 35 < 36 < 143 < 210 < 216 < \cdots.$

(b) $\{p\}: \dfrac{13}{11} < \dfrac{7}{5} < \dfrac{3}{2}.$ (finite).

$$\{l\}: \quad 1 < \frac{13}{11} < \frac{13^2}{11^2} < \frac{7}{5} < \frac{3}{2} < \frac{13^2}{11^3} < \frac{13}{11}\cdot\frac{7}{5} < \cdots.$$

THEOREM 1. *The power of a generalized prime in* $l_n! = l_n l_{n-1} \cdots l_2 l_1$ *is*

$$\left[\frac{l}{p}\right] + \left[\frac{l}{p^2}\right] + \left[\frac{l}{p^3}\right] + \cdots,$$

where $[m]$ *is the number of* l *numbers* $\leqq m$.

Proof. The number of factors of the product $l!$ which are multiples of p, is $[l/p]$; amongst these there are $[l/p^2]$ multiples of p^2, and so on. The sum of all these numbers gives the required power of p. The same reasoning also gives

THEOREM 2. $\phi(p^n) = [p^n] - [p^{n-1}]$, *where* $\phi(m)$ *is the number of* l *numbers less than* m *which are prime to* m.

The general formula for the Euler ϕ function does not hold as $\phi(m)$ is not multiplicative.

THEOREM 3. $\sum_{d|l} \phi(d) = [l]$.

(In this notation $d|l$ if $d \in \{l\}$ and there exists $l_d \in \{l\}$ such that $d \cdot l_d = l$. Call d the greatest common divisor of l_r and l_s, *i.e.*, $(l_r, l_s) = d$ if

(1) d is a common divisor of l_r and l_s

(2) every common divisor of l_r and l_s divides d.)

Proof. Consider the sequence

$$1 = l_1, l_2, \cdots, l. \tag{1}$$

If d is any divisor of l, there are in this sequence $[l/d]$ multiples of d, namely the numbers

$$1 \cdot d, l_2 \cdot d, \cdots l_r \cdot d, \cdots, (l/d) \cdot d. \tag{2}$$

Some of the numbers (2) have a g.c.d. d with l. The g.c.d. of $l_r \cdot d$ with l is d if and only if $(l_r, l/d) = 1$. There are $\phi(l/d)$ numbers $l_r d$ with this property. Since all numbers (1) have a g.c.d. with l,

$$\sum_{d|l} \phi(l/d) = [l].$$

But when d runs through all divisors of l, so does l/d, and the proof is complete.

Multiplicative functions. $\theta(l)$ is multiplicative if it is defined for all l and is not identically zero for any l and if $\theta(l_m l_n) = \theta(l_m)\theta(l_n)$ when $(l_m, l_n) = 1$. If $\theta(l)$ is multiplicative then $\theta(1) = 1$ and also

THEOREM 4. *If* $l = \prod_{j=1}^{k} p_j^{\nu_j}$, then

$$\sum_{d|l} \theta(d) = \prod_{j=1}^{k} \{1 + \theta(p_j) + \cdots + \theta(p_j^{\nu_j})\}.$$

Proof. If $l=1$, the right-hand side is taken to be 1. If $l>1$, the right-hand side is the sum of products of the form

$$\theta(p_1^{\beta_1}) \cdots \theta(p_k^{\beta_k}) = \theta(p_1^{\beta_1} \cdots p_k^{\beta_k})$$

in which all possible products of this form are contained and no product occurs more than once. This is what occurs on the left-hand side.

COROLLARY. *The function l^s is multiplicative and so*

$$\sum_{d|l} d^s = \prod_{j=1}^{k} (1 + p_j^s + \cdots + p_j^{\nu_j s}).$$

THEOREM 5. $\sum_{r=1}^{n} \tau(l_r) = \sum_{h=1}^{n} [l_n/l_h]$, *where $\tau(l)$ is the number of divisors of l.*

Proof. Let $f(l_n)$ denote the right-hand side. Now $[l_n/l_h]=[l_{n-1}/l_h]=1$ or 0 according as l_h is a divisor of l_n or not. Hence

$$\begin{aligned} f(l_n) - f(l_{n-1}) &= \sum_{h=1}^{n}\left[\frac{l_n}{l_h}\right] - \sum_{h=1}^{n-1}\left[\frac{l_{n-1}}{l_h}\right] \\ &= \sum_{h=1}^{n}\left[\frac{l_n}{l_h}\right] - \sum_{h=1}^{n}\left[\frac{l_{n-1}}{l_h}\right] + \left[\frac{l_{n-1}}{l_n}\right] \\ &= \tau(l_n) + 0, \end{aligned}$$

$$\sum_{r=1}^{n} \tau(l_r) = \sum_{r=1}^{n} \{f(l_n) - f(l_{n-1})\} + 1.$$

(Putting $s=0$ and $s=1$ in the corollary to Theorem 4 gives the number of divisors of l and the sum of the divisors of l.)

The Möbius function. Define $\mu(l)=0$ if l has a square factor; $\mu(l)=(-1)^k$, where k denotes the number of prime divisors of l and l has no square factor; $\mu(1)=1$. Then $\mu(l)$ is multiplicative.

THEOREM 6.

$$\sum_{d|l} \mu(d) = \begin{cases} 0 & \text{when } l \neq 1, \\ 1 & \text{when } l = 1. \end{cases}$$

$$\sum_{d|l} \mu(d)/d = \begin{cases} \prod_{j=1}^{k} (1 - p_j^{-1}) & \text{when } l \neq 1, \\ 1 & \text{when } l = 1. \end{cases}$$

These results follow immediately from Theorem 4 by putting $\theta(l)=\mu(l)$ and $\theta(l)=\mu(l)/l$, respectively.

THEOREM 7. (An inversion formula). *If* $G(l_n)=\sum_{d|l_n} F(d)$, then $F(l_n) = \sum_{d|l_n} \mu(l_n/d)\, G(d)$.

Proof.

$$\sum_{d|l_n} \mu(l_n/d)\, G(d) = \sum_{d|l_n} \mu(d)\, G(l_n/d) = \sum_{d|l_n} \mu(d) \sum_{c|(l_n/d)} F(c)$$
$$= \sum_{(cd)|l_n} \mu(d)F(c) = \sum_{c|l_n} F(c) \sum_{d|(l_n/c)} \mu(d).$$

But, from Theorem 6,

$$\sum_{d|(l_n/c)} \mu(d) = 0$$

unless $c = l_n$ when the sum is 1. Hence the theorem is proved.

(Using this theorem with Theorem 3 shows that

$$\phi(l_n) = \sum_{d|l_n} \mu(l_n/d)[d].)$$

References

1. A. Beurling, Analysis of the asymptotic law of the distribution of generalised prime numbers, Acta Math., vol. 68, 1937, pp. 255–291.

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THE 1960 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twenty-first William Lowell Putnam Mathematical Competition held on December 3, 1960, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of the University of California, Berkeley, California. The members of the team were George Bergman, Stefan Burr, and Jon Folkman; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Stephen Adler, M. Alan Landman, and Daniel Quillen; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Stephen Orszag, Frank Rubin, and John Wells; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Michigan State University, East Lansing, Michigan. The members of the team were Richard Freeman, Robert E. Greene, and Ted E. Petrie; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Harold G. Diamond, William B. Easton, and Jeffrey P. Rubens; to each of these a prize of ten dollars is awarded.

The five individuals ranking highest in the examination, named in alphabetical order, are William R. Emerson, California Institute of Technology; Jon Folkman, University of California at Berkeley; M. Hochster, Harvard University; Louis Jaeckel, University of California at Los Angeles; and Samuel Klein, College of the City of New York.

The five individuals ranking second highest in the examination, named in alphabetical order, are Robert Kilmoyer, Lebanon Valley College; John N. Mather, Harvard University; Daniel Quillen, Harvard University; Bill Waterhouse, Harvard University; and John F. Wilkinson, California Institute of Technology.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being Edward A. Bender, Harold M. Stark, and John F. Wilkinson; Polytechnic Institute of Brooklyn, Brooklyn, New York, the members of the team being Burton Fein, George Glauberman, and Errol Pomerance; Reed College, Portland, Oregon, the members of the team being Lynne Barnes, Loyd Hopper, and David Ragozin; University of California at Los Angeles, the members of the team being George Chapline, Louis Jaeckel, and Stanton Philipp; University of Notre Dame, Notre Dame, Indiana, the members of the team being Robert Burckel, William O'Connel, and James Wirth; and Yeshiva University, New York, New York, the members of the team being Stanley Boylan, Jonathan Ginsberg, and Benjamin Weiss.

Honorable mention is given to the following eighteen individuals, named in alphabetical order: Stephen Adler, Harvard University; (Mrs.) Rae H. Alderfer, Ursinus College; George Bergman, University of California at Berkeley; Stefan Burr, University of California at Berkeley; Rufus Clay, Yale University; Larry Dornhoff, University of Nebraska; Joseph Ferrar, Michigan State University; George Glauberman, Polytechnic Institute of Brooklyn; John A. Holbrook, Queen's University (Kingston, Ontario); Anthony Knapp, Dartmouth College; Martin Lampe, Harvard University; M. Alan Landman, Harvard University; John Lindsey, California Institute of Technology; Stephen Orszag, Massachusetts Institute of Technology; David Ragozin, Reed College; Lawrence C. Shepley, Swarthmore College; Stuart Sidney, Yale University; and Benjamin Weiss, Yeshiva University.

A total of eleven hundred and nine contestants from one hundred and sixty-six colleges and universities (one hundred and thirty-seven of these having teams)

entered the Competition. Eight hundred and sixty-seven contestants from one hundred and sixty colleges and universities (one hundred and twenty-eight having teams) participated in the examination on December 3, 1960.

The following is a list, in alphabetical order, of all colleges and universities which entered teams in the Competition: Adams State College, Agricultural and Mechanical College of Texas, American University, Amherst College, Anna Maria College, Arizona State College, Bethel College, Bowdoin College, Brandeis University, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Case Institute of Technology, Central Michigan University, Chatham College, College of the City of New York, College of the Holy Cross, Columbia University, Cornell University, Dakota Wesleyan University, Dana College, Dartmouth College, Delaware State College, Eastern New Mexico University, East Central State College, Fordham University, George Pepperdine College, Georgetown University, Georgia Institute of Technology, Grambling College, Grinnell College, Hamilton College (McMaster University), Harvard University, Haverford College, Humboldt State College, Illinois Institute of Technology, Iowa State University, Kalamazoo College, Kenyon College, King College, Knox College, Lebanon Valley College, Linfield College, Manhattan College, Mankato State College, Marquette University, Marymount College, Massachusetts Institute of Technology, McGill University, Memphis State University, Michigan State University, Monmouth College, Mount Allison University, Mundelein College, Newark College of Engineering, New Mexico State University, Northeastern University, Northern Montana College, North Texas State College, Oberlin College, Occidental College, Ohio State University, Ohio Wesleyan University, Oklahoma Baptist University, Oklahoma State University, Oregon State College, Otterbein College, Polytechnic Institute of Brooklyn, Pomona College, Purdue University, Queens College, Queen's University, Radcliffe College, Reed College, Rockford College, Rockhurst College, Rosary College, Rose Polytechnic Institute, Saint Ambrose College, Saint Francis Xavier University, Saint Martin's College, San Diego State College, San Jose State College, Seattle University, Shepherd College, South Dakota School of Mines, Southeastern State College, Southwestern at Memphis, Stanford University, Swarthmore College, Texas College, Tulane University, University of Alberta, University of Arizona, University of Buffalo, University of British Columbia, University of California (Berkeley), University of California (Los Angeles), University of Colorado, University of Detroit, University of Dallas, University of Illinois, University of Kansas, University of Manitoba, University of Michigan, University of Minnesota, University of Montreal, University of Nebraska, University of Notre Dame, University of Ottawa, University of Puerto Rico, University of Rochester, University of San Francisco, University of Santa Clara, University of Southeastern Louisiana, University of Texas, University of the South, University of Toronto, University of Washington, University of Waterloo, University of Western Ontario, United States Air Force Academy, United States

Naval Academy, Ursinus College, Valparaiso University, Villanova University, Wake Forest College, Washington State University, Washington University, Wayne State University, Wesleyan University, William Jewell College, William Marsh Rice University, Yale University, and Yeshiva University.

The following institutions, in alphabetical order, entered individual contestants only: Albright College, Antioch College, Bethany Nazarene College, Clemson College, College of Saint Catherine, College of Saint Thomas, Eastern Baptist College, Fresno State College, Hofstra College, Hunter College, Idaho State College, Kent State University, Macalester College, Madison College, Montana State College, North Carolina State College, Sacramento State College, Saint Olaf College, Southern University, State University of New York (Long Island Center), University of Arkansas, University of Houston, University of Iowa, University of Kentucky, University of Massachusetts, University of Wyoming, Westminster College, West Virginia University, and Xavier University.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

Those participating in the competition were given the following problems to solve:

Part I

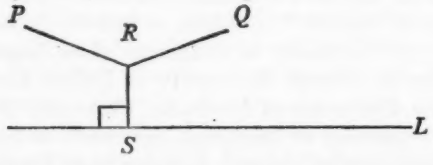
1. Let n be a given positive integer. How many solutions are there in ordered positive integer pairs (x, y) to the equation

$$\frac{xy}{x+y} = n?$$

2. Show that if 3 points are inside a closed square of unit side, then some pair of them are within $\sqrt{6}-\sqrt{2}$ units apart.
3. Show that if t_1, t_2, t_3, t_4, t_5 are real numbers, then

$$\sum_{j=1}^{5} (1 - t_j)e^{\sum_{k=1}^{j} t_k} \leq e^{t_5}.$$

4. Given two points in the plane, P and Q, at fixed distances from a line L, and on the same side of the line, as indicated, the problem is to find a third point R so that $PR+RQ+RS$ is a minimum, where RS is perpendicular to L. Consider all cases.



5. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.
6. A player throwing a die scores as many points as on the top face of the die, and is to play until his score reaches or passes a total n. Denote by $p(n)$ the probability of making exactly the total n, and find the value of $\lim_{n\to\infty} p(n)$.

7. Let $N(n)$ denote the smallest positive integer N such that $x^N = 1$ for every permutation x on n symbols, where 1 denotes the identity permutation. Prove that if $n > 1$,

$$\frac{N(n)}{N(n-1)} = 1 \text{ if } n \text{ is divisible by 2 distinct primes,}$$

$$= p \text{ if } n \text{ is a power of a prime } p.$$

Part II

1. Find all solutions of $n^m = m^n$ in integers n and m $(n \neq m)$. Prove that you have obtained all of them.
2. Evaluate the double series

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-3k-j-(k+j)^2}.$$

3. The motion of the particles of a fluid in the plane is specified by the following components of velocity

$$\frac{dx}{dt} = y + 2x(1 - x^2 - y^2), \qquad \frac{dy}{dt} = -x.$$

Sketch the shape of the trajectories near the origin. Discuss what happens to an individual particle as $t \to +\infty$, and justify your conclusion.

4. Consider the arithmetic progression a, $a+d$, $a+2d$, $\cdots$, where a and d are positive integers. For any positive integer k, prove that the progression has either no exact kth powers, or infinitely many.
5. Define a sequence as follows:

$$a_0 = 0, \qquad a_1 = 1 + \sin(-1), \cdots, a_n = 1 + \sin(a_{n-1} - 1) \cdots.$$

Evaluate

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k.$$

6. Any positive integer may be written in the form $n = 2^k(2l+1)$. Let $a_n = e^{-k}$ and $b_n = a_1 a_2 \cdots a_n$. Prove that $\sum b_n$ converges.
7. Let $g(t)$ and $h(t)$ be real, continuous functions for $t \geqq 0$. Show that any function $v(t)$ satisfying the differential inequality

$$\frac{dv}{dt} + g(t)v \geqq h(t), \qquad v(0) = c,$$

satisfies the further inequality $v(t) \geqq u(t)$, where

$$\frac{du}{dt} + g(t)u = h(t), \qquad u(0) = c.$$

From this, conclude that for sufficiently small $t > 0$, the solution of

$$\frac{dv}{dt} + g(t)v = v^2, \qquad v(0) = c_1$$

may be written

$$v = \max_{w} \left[c_1 e^{-\int_0^t [g(s) - 2w(s)]\,ds} - \int_0^t e^{-\int_s^t [g(s_1) - 2w(s_1)]\,ds_1} w^2(s)\,ds \right],$$

where the maximization is over all continuous functions $w(t)$ defined over some t-interval $[0, t_0]$.

Solutions.* Part I

The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants.

1. $x=n$ does not furnish a solution and therefore $y=(xn)/(x-n)$ is an equivalent equation. (x, y) is a solution of the desired type of this equation if and only if

$$(x, y) = \left(n + i, \frac{n^2}{i} + n\right),$$

where i is a positive integral divisor of n^2.

2. If the points are collinear then two of them must be within $\frac{1}{2}\sqrt{2}$ units. If the points are not collinear a closed triangle with vertices on the boundary of the square and containing these points will have sides not less than the minimum distance between these points. If two vertices are on the same side of the square they will be within 1 unit. If two vertices are on opposite sides of the square a parallel motion of the side joining them until one vertex coincides with a vertex of the square yields a triangle not having smaller sides. Such a triangle will have its short side a maximum provided it is equilateral with side $\sqrt{6}-\sqrt{2}$ units.

3. Let

$$f(t_1, t_2, t_3, t_4, t_5) = \sum_{j=1}^{5} (1 - t_j) \exp\left[\sum_{k=1}^{j} t_k\right]$$

and let f_i denote the partial derivative of f with respect to t_i. The solution of the system $f_i=0$ $(i=1, \cdots, 5)$ yields

$$(*) \qquad f(e^{e^e}, e^e, e, 1, 0) = e^{e^{e^e}}$$

as the only possible extremum of f. Since $f\to-\infty$ as each $t_i\to\infty$ and $f\to 0$ as each $t_i\to-\infty$, it follows that (*) is a maximum value of f.

4. Choose the coordinate system so that the x-axis coincides with L and the points P and Q have coordinates $(-a, p)$ and (a, q), respectively. Without loss of generality it can be assumed that $0<p\leqq q$. To solve the problem we need to find coordinates (x, y) of R such that the function

$$F(x, y) = |y| + \sqrt{\{(x + a)^2 + (y - p)^2\}} + \sqrt{\{(x - a)^2 + (y - q)^2\}}$$

is a minimum. Consideration of $F_x=0$, $F_y=0$ yields $x=\frac{1}{2}\sqrt{3}(p-q)$, $y=\frac{1}{2}(p+q)-\frac{1}{3}\sqrt{3}a$ as a solution provided that $(q-p)/(2a)\leqq\frac{1}{3}\sqrt{3}$. Otherwise $x=-a$, $y=p$ furnishes the minimum.

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper graders will enter into any correspondence concerning them.

5. Let $g(x)=x+h$ so that $f(x+h)=f(x)+h$ and $\{f(x+h)-f(x)\}/h=1$ for all $h\neq 0$. Therefore $f'(x)=1$ and $f(x)=x+c$. Letting $g(x)$ be the zero polynomial shows that it is necessary that $c=0$ and thus $f(x)=x$. It is easily seen that $f(x)=x$ is sufficient.

6. Let $p(n)=\frac{1}{6}\{p(n-1)+p(n-2)+p(n-3)+p(n-4)+p(n-5)+p(n-6)\}$ where $p(0)=1$ and $p(n)=0$ for negative n. Using this in the generating function equation $P(s)=p(0)+p(1)s+\cdots+p(n)s^n+\cdots$ after multiplication of both sides by $(6-s-s^2-s^3-s^4-s^5-s^6)$, we obtain $P(s)=6/(6-s-s^2-s^3-s^4-s^5-s^6)$. This can be expressed as

$$P(s)=\frac{2/7}{1-s}+\frac{2(15+10s+6s^2+3s^3+s^4)}{7(6+5s+4s^2+3s^3+2s^4+s^5)}.$$

Expanding the two fractions into power series we obtain from the coefficients of $P(s)$ that $p(n)=2/7+f_n$ where the f_n are coefficients in the expansion of the second fraction. It can be shown that $f_n\to 0$ and therefore that $\lim p(n)=2/7$.

7. The symmetric group S_n has order $n!$ and contains elements of orders $1,\cdots,n$. Therefore $N(n)=LCM(1,\cdots,n)$. If n is divisible by two distinct primes, then $N(n)=\mathrm{LCM}(1,\cdots,n-1)=N(n-1)$. If n is a power of a prime p, then $N(n)=p\ \mathrm{LCM}(1,\cdots,n-1)=pN(n-1)$.

Solutions. Part II

1. It is clear that neither m nor n may be zero, and it is also readily verified that neither m nor n may equal $+1$ or -1 without having $m=n$. If m and n are both positive, the equation $n^m=m^n$ is equivalent to $(\log n)/n=(\log m)/m$. Now for $x>1$, $(\log x)/x$ is positive, increasing for $1<x<e$, and decreasing for $x>e$. Hence if $(\log n)/n=(\log m)/m$ is to be satisfied, with $m\neq n$, one of m and n must be between 1 and e, while the other is greater than e. Since 2 is the only integer between 1 and e, one of m and n must be 2, and it is immediately seen that this leads to a solution $2^4=4^2$, which must be the only solution if both m and n are positive and $m\neq n$. It is not possible for m and n to be of opposite sign, since the absolute value of one of m^n and n^m would be less than one while the other is greater than one. There is no solution with m and n both negative and of opposite parity since then m^n and n^m have opposite sign. If m and n are both negative and of the same parity, $m^n=n^m$ is equivalent to $(-m)^{-n}=(-n)^{-m}$, so that the only solution with m and n both negative is $(-2)^{-4}=(-4)^{-2}$, since there is only one positive solution. Thus the only solutions are $2^4=4^2$ and $(-2)^{-4}=(-4)^{-2}$.

2. If the given double series is written out it is seen that every term of the geometric progression $\sum_{n=0}^{\infty}2^{-2n}$ occurs once and only once. The double series is positive and convergent, hence absolutely convergent, so the rearrangement does not alter the sum. The sum of the double series is therefore 4/3.

3. The last letter in the first equation is y although broken type made it appear to be v, and it was considered to be y by a majority of those attempting the problem. The behavior of the trajectories near the origin is determined from the linear terms, *i.e.* from the system $dx/dt=y+2x$, $dy/dt=-x$. This system can be solved explicitly. The trajectories are the curves

$$\frac{x}{x+y}=\log|x+y|+C,$$

all of which go to the origin tangent to the line $x+y=0$. The direction of increasing t is away from the origin. To find the behavior as $t\to+\infty$, introduce polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$. It is found that $(d/dt)r^2=4r^2\cos^2\theta(1-r^2)$ so that r^2 is nondecreasing when $r^2<1$ and r^2 is nonincreasing when $r^2>1$. Thus for any solution, the polar coordinate r approaches a limit which must be 1, since dr/dt is different from zero somewhere on every circle $r=$constant, where $r\neq 1$. Thus the solution curves all spiral outward or inward to the circle $r=1$.

If the last letter in the first equation is taken to be a constant v the same procedure will hold at the origin. The linear system is $dx/dt=y+2x(1-v^2)$ $dy/dt=-x$, and so long as $v^2\neq 1$, the complete system corresponds to the linear system. It is found that for $v^2>1$, all trajectories go toward the origin, while if $v^2<1$, all trajectories go away from the origin. When $v^2=1$, the original system is $dx/dt=-2x^3+y$, $dy/dt=-x$, and we find $(d/dt)r^2=-4x^4$, which implies that all trajectories go to the origin. As $t\to+\infty$, we have all trajectories going to the origin if $v^2\geqq 1$. If $v^2<1$, it can be shown that there is a closed curve surrounding the origin to which all trajectories converge.

4. Suppose that an integer x is such that $x^k\equiv a \pmod d$, *i.e.* x^k is a member of the given progression. Then for any positive integer n, $(x+nd)^k\equiv x^k \pmod d$ $\equiv a \pmod d$, so that $(x+nd)^k$ is also in the given progression. Thus, if the progression contains one kth power it contains infinitely many. There are arithmetic progressions which contain no kth powers. *E.g.*, the progression $2, 5, 8, \cdots$ contains no squares (or any even powers).

5. Let $b_n=a_n-1$. Then $b_0=-1$ and $b_n=\sin b_{n-1}$. By induction it follows that every b_n is negative, and that the b_n form an increasing sequence. Hence they have a limit L which must satisfy $L=\sin L$. Thus $L=0$, $b_n\to 0$, and $a_n\to 1$. It is well known that if a sequence $\{a_n\}$ has a limit, the sequence of averages $\{n^{-1}\sum_1^n a_k\}$ has the same limit. Hence the required limit is 1.

6. The problem means that every positive integer can be written uniquely in the form $2^k(2l+1)$, where k and l are nonnegative integers. Thus $k=0$ when n is odd, and k is a positive integer when n is even. Hence $a_n=1$ when n is odd, and $a_n\leqq e^{-1}$ when n is even. Therefore $b_{2n}=b_{2n+1}$ and the series is equivalent to $b_1+2b_2+2b_4+\cdots$. But since $b_{2n+2}/b_{2n}=a_{2n+1}a_{2n+2}\leqq e^{-1}$, the latter series converges. Since the terms approach zero, the grouping of the given series does not affect convergence and the given series converges.

7. Let $p(t)=v(t)-u(t)$. At $t=0$, $p(0)=v(0)-u(0)=c-c=0$. Moreover, subtracting the relation for u from that for v, we find that $(dp/dt)+g(t)p\geqq 0$ or, multiplying by $\exp\{\int_0^t g(s)ds\}$ that

$$d/dt\left[p(t)\ \exp\ \left\{\int_0^t g(s)ds\right\}\right]=0.$$

Integrating and noting that $p(0)=0$, we have for $t\geqq 0$, $p(t)\ \exp\{\int_0^t g(s)ds\}\geqq 0$. But, since the exponential is positive, we have $p(t)\geqq 0$ for $t\geqq 0$, so that $v(t)\geqq u(t)$ for $t\geqq 0$.

Now let $w(t)$ be any continuous function on $t\geqq 0$. Let $g_1(t)=g(t)-2w(t)$ and let $h(t)=-\{w(t)\}^2$. Then, if v satisfies $(dv/dt)+g(t)v=v^2$ with $v(0)=c_1$,

$$(dv/dt)+g_1(t)v=v^2-2wv=(v-w)^2-w^2\geqq -w^2=h(t)$$

with equality if and only if w is chosen to be exactly the solution v. But, if $u'+g_1(t)u=h(t)$ with $u(0)=c_1$, we can solve for u and obtain

$$u(t)=c_1\ \exp\ \left\{-\int_0^t[g(s)-2w(s)]ds\right\}$$
$$-\int_0^t\ \exp\ \left\{-\int_s^t[g(s_1)-2w(s_1)]ds_1\right\}w^2(s)ds.$$

Now by the first part, $v(t)\geqq u(t)$ with equality if and only if $w(t)$ is chosen equal to $v(t)$. Thus $v(t)$ is the maximum of all possible $u(t)$ as $w(t)$ ranges over all continuous functions.

CORRECTIONS

In the paper *The wedge product* by Gerald Berman (this MONTHLY, vol. 68, 1961, pp. 112–119), there is an omission following (2.2) on page 112. This should read "where the summation is taken over all values of the μ's and γ's such that $\mu_i+\gamma_i\equiv\epsilon_i$ (mod 2), $i=1,\cdots,n$, and $\cdots$." Again, in (2.5), the sums $\mu_i+\gamma_i$ are taken mod 2. Finally, the * mapping in Section 3 is *not* an automorphism, but it is useful in describing the algebra.

In the paper *A generalized Fibonacci sequence* by A. S. Horadam (this MONTHLY, vol. 68, 1961, pp. 455–459), equation (17) on page 457 is incorrect. It should read

$$(17)\qquad \frac{H_{n+r}+(-1)^rH_{n-r}}{H_n}=F_{r+1}+F_{r-1}=a^r+b^r.$$

MATHEMATICAL NOTES

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Material for this department should be sent to Roy Dubisch, Department of Mathematics, University of Washington, Seattle 5, Washington.

A FUNCTIONAL INEQUALITY*

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Ewing and Utz [2] determined all real-valued continuous functions which satisfy the functional equation

$$f^n(x) = f(x) \qquad -\infty < x < \infty, \tag{1}$$

where f^n denotes the nth iterate of f and n is a given positive integer. Donald I. Kurth and I [1] solved the following functional inequality

$$f^n(x) \geqq x, \qquad -\infty < x < \infty. \tag{2}$$

In this note I shall study a more general functional inequality.

Throughout the paper $\phi(x)$ denotes a continuous strictly increasing function such that $\lim_{x\to\infty}\phi(x) = \infty$, $\lim_{x\to-\infty}\phi(x) = -\infty$ and $f(x)$ denotes a continuous function of x. I shall obtain some properties of the solutions of the functional inequality

$$f^n(x) \geqq \phi(x) \qquad -\infty < x < \infty. \tag{3}$$

THEOREM 1. *If $f(x)$ is a solution of* (3), *then* $\lim_{x\to\infty} f(x) = \infty$ *or* $\lim_{x\to\infty} f(x) = -\infty$.

Proof. If the theorem is false, there exists a number a, and a sequence $\{x_j\}$ tending to infinity such that $f(x_j)\to a$. For each value of n, $\lim_{j\to\infty} f^n(x_j) = f^{n-1}(a)$, while x_j tends to infinity. This situation is impossible if $f(x)$ is a solution of (3).

THEOREM 2. *If $f(x)$ is a solution of* (3) *and* $\lim_{x\to\infty} f(x) = -\infty$, *then* $\lim_{x\to-\infty} f(x) = \infty$.

Proof. If not, there exists a number $b \geqq -\infty$ and a sequence $\{y_j\}$ tending to minus infinity such that $f(y_j)$ tends to b. Since $f(x)$ tends to minus infinity as x tends to infinity, for sufficiently large values of j there exists a point x_j such that $f(x_j) = y_j$ and x_j tends to infinity. Then $f^2(x_j)$ tends to b and for $n > 2$, $f^n(x_j)$ tends to $f^{n-2}(b)$ as j tends to infinity. On the other hand $\phi(x_j)$ tends to infinity and we have the same type of contradiction as in Theorem 1.

THEOREM 3. *If $f(x)$ is a solution of* (3) *for an odd value of n, then* $\lim_{x\to\infty} f(x) = \infty$.

Proof. If not, $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$. There exists a fixed-point c of the mapping f. The interval $[c, \infty)$ is mapped onto an interval con-

* This work was supported by Nonr 710(16).

taining $(-\infty, c]$ by f, and the interval $(-\infty, c]$ is mapped onto an interval containing $[c, \infty)$. It is easy to see that all odd iterates of f map the interval $[c, \infty)$ onto an interval containing $(-\infty, c]$. Since $\phi(x)$ tends to infinity as x tends to infinity, f cannot be a solution of (3) for an odd value of n.

LEMMA. *If* $\limsup_{x\to\infty} f(x) = \infty$ *and* x_0 *is a point such that* $f(x_0) < x_0$, *then, for each integer* r, *there is a point* $x_r \geqq x_0$ *such that* $f^r(x_r) = x_0$.

This lemma is proved rather easily by use of finite induction and the intermediate value theorem.

THEOREM 4. *If* $f(x)$ *is a solution of* (3) *such that* $f(x)$ *tends to infinity as* x *tends to infinity, then* $f(x) \geqq \phi(x)$ *whenever* $f(x) < x$. *In particular this holds if* $f(x)$ *is a solution of* (3) *for an odd value of* n.

Proof. Suppose x_0 is such that $f(x_0) < x_0$, $f(x_0) < \phi(x_0)$. By the preceding lemma, there exists a point x_{n-1} such that $x_{n-1} > x_0$, $f^{n-1}(x_{n-1}) = x_0$. Thus $f^n(x_{n-1}) = f(x_0) < \phi(x_0) < \phi(x_{n-1})$, and f is not a solution of (3).

COROLLARY. *If* $f(x)$ *is a solution of*

$$f^n(x) \geqq x \tag{4}$$

for an odd value of n, *then* $f(x) \geqq x$.

LEMMA. *If* $f(x)$ *satisfies a Lipschitz condition, with Lipschitz constant* L,

$$|f^n(x) - x| \leqq \left(\sum_{k=1}^{n} L^k\right) |f(x) - x|.$$

Proof. We have $|f^2(x) - f(x)| = |f(f(x)) - f(x)| \leqq L|f(x) - x|$ and $|f^3(x) - f^2(x)| = |f(f^2(x)) - f(f(x))| \leqq L^2|f(x) - x|$. More generally $|f^k(x) - f^{k-1}(x)| \leqq L^{k-1}|f(x) - x|$. Hence

$$|f^n(x) - x| \leqq \sum_{k=1}^{n} |f^k(x) - f^{k-1}(x)| \leqq \left(\sum_{k=1}^{n} L^k\right) |f(x) - x|.$$

THEOREM 5. *If* $f(x)$ *is a solution of* (3), *for an odd value of* n, *and it satisfies a Lipschitz condition with Lipschitz constant* L, *and*

$$m = \inf_{\phi(x) > x} [(f(x) - x)]/[(\phi(x) - x)],$$

then

$$n > \log[(1 + (L - 1)/m)]\log L \quad \text{for } L \neq 1 \text{ and } n > 1/m \quad \text{for } L = 1.$$

In particular if $m = 0$, *then* $f(x)$ *is not a solution of* (3) *for any odd value of* n.

Note: It follows from Theorem 1 that $m \geqq 0$.

Proof. If ϵ is a positive number and x is a point such that $\phi(x) > x$, then

$0<f(x)-x<(m+\epsilon)(\phi(x)-x)$. From the preceding lemma we have

$$f^n(x) - x \leqq \sum_{k=1}^{n} L^k[f(x) - x] \leqq \sum_{k=1}^{n} L^k(m + \epsilon)(\phi(x) - x).$$

If $L \neq 1$,

$$f^n(x) - x \leqq \left(\sum_{k=1}^{n} L^k \right) (m + \epsilon)(\phi(x) - x).$$

If $f(x)$ is a solution of (3) then, since ϵ is arbitrary,

$$m \sum_{k=1}^{n} L^k = m(L^n - L)/(L - 1) < 1,$$

and thus $n > [\log[1+(L-1)/m]]/[\log L]$. If $L=1$, we show in a similar manner that $n > 1/m$.

If the requirement that $f(x)$ satisfy a Lipschitz condition is removed, then the theorem is no longer true. For example let $\phi(x)=x^{1/5}$ if $0 \leqq x \leqq 1$ and $\phi(x)=x$ if $x<0$, or $x>1$. Then, let $f(x)=x^{1/3}$ if $0 \leqq x \leqq 1$ and $f(x)=x$ if $x<0$ or $x>1$. Then $f^n(x) \geqq \phi(x)$ for $n \geqq 2$ despite the fact that $(f(x)-x)/(\phi(x)-x)$ approaches zero as x approaches zero through positive values.

THEOREM 6. *If $\phi(x)-x$ is nondecreasing on the set $\{x|\phi(x)>x\}$, then the following conditions ensure that $f(x)$ is a solution of* (3) *for $n > [1/m]$:*

(i) $$f(x) \geqq \phi(\beta)$$

whenever $f(x)<x$, where $\beta = \sup\{x|f(x)<x\}$,

(ii) $$\inf [f(x) - x)]/[\phi(x) - x] \geqq m > 0,$$

where the infinum is taken over all x such that $\phi(x)>x$.

Proof. We note first that if x' is a point such that $\phi(x') \leqq x'$ then our assumptions insure that $f(x') \geqq \phi(x')$. Let n_0 denote the least positive integer such that $f^{n_0}(x')<\phi(x')$. Then $f^{n_0-1}(x') \geqq \phi(x')$, so that $f(f^{n_0-1}(x'))<f^{n_0-1}(x')$; hence $f^{n_0-1}(x')<\beta$. By (i) we have $f^{n_0}(x')=f[f^{n_0-1}(x')]>\phi(\beta)$. If $\beta<x'$, $f(x')>x'$ and it is easy to show that $f^n(x') \geqq x' \geqq \phi(x')$ for all n. If $\beta \geqq x'$, then $f^{n_0}(x')>\phi(\beta)>\phi(x')$, since ϕ is increasing; we thus have a contradiction and we may conclude that $f^n(x') \geqq \phi(x')$ for all positive integers n and all points x' such that $\phi(x') \leqq x'$.

It remains to deal with points x'' such that $\phi(x'')>x''$.

If for some integer n_0 we have $f^{n_0}(x'')>\phi(f^{n_0}(x''))$, then, by the same kind of induction as was used earlier in the proof, we may show that $f^n(x'') \geqq \phi(x'')$ for all $n \geqq n_0$. Thus we need only consider points x'' such that $f^k(x'')<\phi(f^k(x''))$ $(k \leqq [1/m])$, that is, points x'' such that for $k \leqq [1/m]$ the point $f^k(x'')$ belongs to the set $\{y|\phi(y) > y\}$. For such points we have $f^k(x'') - f^{k-1}(x'') \geqq m[\phi(f^{k-1}(x'') - f^{k-1}(x'')]$. The point x'' belongs to the set $\{y|\phi(y)>y\}$, and

since $\phi(y)-y$ is nondecreasing on this set, all points to the right of x'' belong to it. In particular, since $f^{k-1}(x'')>x''$ for $0<k<[1/m]$, by our assumption on ϕ and the above equation, $f^k(x'')-f^{k-1}(x'')\geqq m[\phi(x'')-x'']$. If $n>[1/m]$,

$$f^n(x) - x'' = \sum_{k=1}^{n} f^k(x'') - f^{k-1}(x'') \geqq nm[\phi(x'') - x''] \geqq \phi(x'') - x'',$$

so that $f^n(x'')\geqq\phi x''$. This completes the proof of the theorem.

We now deal with the case where $f(x)$ is a solution of (3) for an even value of n.

THEOREM 7. *If $f(x)$ is a solution of* (3) *for an even value of n, then $f^2(x)\geqq\phi(x)$ whenever $f^2(x)<x$.*

Proof. It follows from Theorems 1 and 2 that $f^2(x)$ tends to infinity as x tends to infinity. Theorem 7 now follows from Theorem 5.

For the inequality

$$f^n(x) \geqq x \tag{4}$$

we have more specific information.

THEOREM 8. *If $f(x)$ is a solution of* (4) *for an even value of n then either $f(x)\geqq x$ or $f^2(x)=x$.*

Proof. If there exists a point x such that $f(x)<x$, then $\lim_{x\to-\infty} f(x)=\infty$, $\lim_{x\to\infty} f(x)=-\infty$. There exists a fixed point c of the mapping f. If $x>c$, then $f(x)\neq c$ for otherwise there would exist a point $x_1>c$ such that $f^k(x_1)=c<x_1$ for $k\geqq 2$. Hence for $x>c$ we have either $f(x)>c$ or $f(x)<c$. In the former case $\lim_{x\to\infty} f(x)=\infty$ and $f(x)>x$ for all values of x. In the latter case it is easy to see that $f(x)>c$ for $x<c$. If this were not true, there would exist a point $y_2<c$, such that $f(y_2)\leqq c$. Since $\lim_{x\to\infty} f(x)=-\infty$ in the case we are considering, there exists a point $x_2>c$ such that $f(x_2)=y_2$. Hence $f^2(x_2)\leqq c<x_2$. This is impossible by Theorem 7. Hence $f(x)>c$ for $x<c$.

Now let M_1 be a number greater than c. The function f maps the interval $[c, M_1]$ into an interval $[m_1, c]$ (note that $m_1<c$). This interval is mapped by f onto an interval $[c, M_2]$ with $M_2>c$ and the latter interval is mapped by f onto an interval $[m_2, c]$ with $m_2<c$. We will show that $M_1=M_2$.

We have

$$M_2 = \max_{c\leqq x\leqq M_1} f^2(x) \geqq f^2(M_1) \geqq M_1.$$

If $M_1<M_2$, then $[c, M_1]\subset[c, M_2]$ and $[m_1, c]\subseteq[m_2, c]$; that is $m_1\geqq m_2$. But since

$$m_2 = \min_{m_1\leqq x\leqq c} f^2(x),$$

there exists a point x_1 such that $m_1\leqq x_1<c$ and $f^2(x_1)=m_2\leqq m_1\leqq x_1$. Since $f(x)$ is a solution of (4) we must have $m_1=m_2$ and consequently $M_1=M_2$ or $f^2(M_1)=M_1$.

Since M_1 was chosen arbitrarily we have $f^2(x)=x$ for $x>c$. Similarly one can show that $f^2(x)=x$ for $x<c$. Hence $f^2(x)$ is identically equal to x. This completes the proof.

I wish to thank Professor Seymour Schuster and my colleague Mr. William A. Dolid for suggesting the problem.

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2. G. M. Ewing and W. R. Utz, A functional equation, Canad. J. Math., vol. 5, 1953, pp. 101–103.

A BOUND FOR THE SOLUTION OF A LINEAR EQUATION

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The object of the present note is to establish a bound for solutions of the equation $\ddot{y}+f(t)y=0$, assuming that $f(t)$ is continuous and that $0<f_1\leqq f(t)\leqq f_2$.

The case $y_0>0$, $\dot{y}_0>0$ is considered first. For $y>0$, $\dot{y}>0$, it follows that

$$-2f_2y\dot{y}\leqq 2\dot{y}\ddot{y}\leqq -2f_1y\dot{y}. \tag{1}$$

Integrating from 0 to t, we obtain $\dot{y}_0^2-f_2(y^2-y_0^2)\leqq\dot{y}^2\leqq\dot{y}_0^2-f_1(y^2-y_0^2)$. Noting the zeros of the left- and right-hand sides of this inequality, and that $\ddot{y}$ is bounded away from zero if y is bounded away from zero, we find the following inequality for the first maximum M occurring to the right of $t=0$:

$$M_1\equiv\sqrt{(y_0^2+\dot{y}_0^2/f_2)}\leqq M\leqq\sqrt{(y_0^2+\dot{y}_0^2/f_1)}\equiv M_2. \tag{2}$$

Supposing this maximum occurs at $t=t'$, integration of (1) from t to t', where $0\leqq t\leqq t'$, yields $-f_2(M^2-y^2)\leqq-\dot{y}^2\leqq-f_1(M^2-y^2)$, from which we obtain

$$\int_0^{t'}\sqrt{f_2}dt\geqq\int_{y_0}^{M}\frac{dy}{\sqrt{(M^2-y^2)}}\geqq\int_0^{t'}\sqrt{f_1}dt,$$

and thus that

$$[\tfrac{1}{2}\pi-\arcsin(y_0/M)]/\sqrt{f_2}\leqq t'\leqq[\tfrac{1}{2}\pi-\arcsin(y_0/M)]/\sqrt{f_1}.$$

To investigate the behavior of y between $t=t'$ and the first zero of y to the right of t', occurring at $t=T$, we begin with the inequality $-2f_1y\dot{y}\leqq2\dot{y}\ddot{y}\leqq-2f_2y\dot{y}$, and integrate from t' to t, where $t'\leqq t\leqq T$, to obtain

$$-f_1(y^2-M^2)\leqq\dot{y}^2\leqq-f_2(y^2-M^2). \tag{3}$$

Hence

$$\int_{t'}^{T}-\sqrt{f_2}\,dt\leqq\int_{M}^{0}\frac{dy}{\sqrt{(M^2-y^2)}}\leqq\int_{t'}^{T}-\sqrt{f_1}\,dt,$$

implying in turn that

$$(\tfrac{1}{2}\pi)/\sqrt{f_2} \leqq T - t' \leqq (\tfrac{1}{2}\pi)/\sqrt{f_1},$$

and thus that

$$[\pi - \arcsin(y_0/M)]/\sqrt{f_2} \leqq T \leqq [\pi - \arcsin(y_0/M)]/\sqrt{f_1}.$$

Since $\dot{y}<0$ for $y>0$, the maximum of $|\dot{y}|$ over $[0, T]$ occurs at either $t=0$ or $t=T$. Inequality (2) implies $|\dot{y}_0| \leqq \sqrt{(f_2)}M$, while (3) yields

$$\sqrt{f_1}\,M \leqq |\dot{y}_T| \leqq \sqrt{f_2}\,M.$$

Thus $\sqrt{(f_2)}M$ furnishes an upper bound for $|\dot{y}|$ over $[0, T]$.

If $y_0>0$ and $\dot{y}_0<0$, we obtain, proceeding similarly,

$$[\arcsin(y_0/M_1)]/\sqrt{f_2} \leqq T \leqq [\arcsin(y_0/M_2)]/\sqrt{f_1},$$

and,

$$\sqrt{(f_1)}\,M_2 \leqq |\dot{y}_T| \leqq \sqrt{(f_2)}\,M_1.$$

Since $M_1 \leqq M \leqq M_2$, we can alter these inequalities to resemble the ones obtained previously:

$$[\arcsin(y_0/M)]/\sqrt{f_2} \leqq T \leqq [\arcsin(y_0/M)]/\sqrt{f_1},\quad \sqrt{(f_1)}\,M \leqq |\dot{y}_T| \leqq \sqrt{(f_2)}\,M.$$

Exactly similar results are obtained for $y_0<0$.

The simplest results follow for $y_0=0$. In this case, beginning the first interval with a slope of $|\dot{y}_0|$, we have an upper bound of $|\dot{y}_0|/\sqrt{f_1} \equiv a$ for $|y|$ over the first interval, an upper bound of $\sqrt{(f_2)}M \leqq |\dot{y}_0|\sqrt{f_2/f_1} \equiv |\dot{y}_0|r$ for $|\dot{y}|$, and a lower bound of $\pi/\sqrt{f_2} \equiv b$ for the interval width. These conditions give rise to an upper bound for the greatest possible variation of y. Thus, for the second interval, we begin with a slope in absolute value not greater than $|\dot{y}_0|r$, obtain an upper bound for $|y|$ of $|\dot{y}_0|r/\sqrt{f_1} = ar$, and end with a slope in absolute value not greater than $\sqrt{f_2}(|\dot{y}_0|r/\sqrt{f_1}) = |\dot{y}_0|r^2$, etc. Thus $|y|$ will be bounded by the expression

$$ar^{t/b} = [|\dot{y}_0|/\sqrt{f_1}][\sqrt{(f_2/f_1)}]^{\sqrt{(f_2)}t/\pi} = Be^{k't},$$

which has values of $a, ar, \cdots$, at $t = 0, b, \cdots$, where $B = |\dot{y}_0|/\sqrt{f_1}$, $k' = (\sqrt{f_2}/2\pi)\ln(f_2/f_1)$.

If we begin with $y_0 \neq 0$, then the first interval must be treated separately. It would seem simplest to state a bound over the first interval of width b', where $b' = [\pi - \arcsin(y_0/M)]/\sqrt{f_2}$, corresponding to $y_0 > 0$, $\dot{y}_0 > 0$, or $b' = [\arcsin(y_0/M)]/\sqrt{f_2}$, corresponding to $y_0>0$, $\dot{y}_0<0$. For these cases, we take $\sqrt{(y_0^2+\dot{y}_0^2/f_1)}$ as an upper bound of M, and $\sqrt{f_2}$ times this quantity as an upper bound for $|\dot{y}|$, over the first interval. The only modification then of the previous bound, beyond an axis shift, for $t \geqq b'$ is the replacement of $|\dot{y}_0|$ by $\sqrt{(\dot{y}_0^2+f_1y_0^2)}$.

Loud [1] obtained a bound on $|y|$ of the form Ae^{kt}, where k is equal to $(f_2-f_1)/[2\sqrt{2}\sqrt{(f_2+f_1)}]$, established assuming only that $3f_1+f_2 \geqq 0$. For

certain values of the constants, the result just obtained is somewhat better; *e.g.*, for $f_1=9$, $f_2=16$, we find that $k=0.495$ and $k'=0.366$, approximately, or for $f_2/f_1=1+\epsilon$, where $\epsilon>0$ is small, k/k' is approximately $\pi/\sqrt{2}$.

To explicitly illustrate the type of unbounded behavior inferred by the previous discussion, it is simplest to consider an example in which $f(t)$ is piecewise continuous. Let $f(t)$ be of period $\frac{3}{4}\pi$, and

$$f(t) = 1,\ 0 \leqq t < \tfrac{1}{2}\pi; \qquad f(t) = 4,\ \tfrac{1}{2}\pi \leqq t < \tfrac{3}{4}\pi.$$

The basic equation has a unique continuous solution for functions of this type, and the previously established bound is valid.

Assuming $y(0)=0$, $\dot{y}(0)=\dot{y}_0>0$, we obtain

$$\begin{aligned} y(t) &= \dot{y}_0 \sin t, && 0 \leqq t < \tfrac{1}{2}\pi, \\ y(t) &= \dot{y}_0 \cos 2(t-\tfrac{1}{2}\pi), && \tfrac{1}{2}\pi \leqq t < \tfrac{3}{4}\pi, \\ y(t) &= -2\dot{y}_0 \sin (t-\pi), && \tfrac{3}{4}\pi \leqq t < \tfrac{5}{4}\pi, \end{aligned}$$

etc. Thus, beginning with a slope of $\dot{y}_0$, $y(t)$ attains a maximum of $\dot{y}_0$ on the first interval of width $\frac{3}{4}\pi$ and its slope approaches $-2\dot{y}_0$ as t approaches $\frac{3}{4}\pi$. These numbers will be multiplied by -2 for the second interval, etc. A bounding curve for this example could be $|\dot{y}_0|\,2^{t/b}$, where $b=\frac{3}{4}\pi$, and, if we shift this curve $\frac{1}{2}\pi$ units to the right, the maxima of $|y(t)|$ actually lie on the resulting curve.

Perhaps the best-known example is furnished by the Mathieu equation: $\ddot{y}+(a+b\cos 2t)y=0$. For $a>|b|>0$, it is of the type under consideration here. Letting y_1 and y_2 represent a pair of independent solutions, Ince ([2], pp. 175–178) shows that for certain values of a and b, y_1 may be chosen as a periodic function. In such a case y_2 is then aperiodic and of the form $c_1tp_1(t)+c_2p_2(t)$, where $p_1(t)$ and $p_2(t)$ are periodic functions representable by series of Fourier type. Thus a bound for $|y(t)|$ would be of the form $At+B$.

The derivation of the inequalities for M, T, and $|\dot{y}_T|$ was suggested by similar derivations established by Taam [3] for the equation $\ddot{x}+p(t)x+2q(t)x^3=F(t)$. Assuming that p, q, and F have a common period L, p and q are even while F is odd, and $0<p_1\leqq p\leqq p_2$, $0<q_1\leqq q\leqq q_2$ for all t, and $0\leqq F_1\leqq F\leqq F_2$ over $(0, \frac{1}{2}L)$, he established a variety of facts concerning the existence and behavior of harmonic and subharmonic solutions. A careful study of his derivations indicates that, apart from the corollaries, all the arguments and conclusions seem to be valid if $p_1=0$, or even if $p(t)\equiv 0$. Thus his paper seems to yield results which complement those obtained by Morris [4, 5] for the equation $\ddot{x}+2x^3=e(t)$ where it was assumed that the forcing term was even and of period 2π and it was established that this equation yields a profusion of subharmonic solutions.

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SOME SPECIAL INTEGRALS

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1. In 1957 the following problem was published by Bremekamp*: "Show that

$$\int_0^{\pi} (\log \sin \phi)^2 d\phi = \frac{1}{12}\pi^3 + \pi(\log 2)^2."$$

Some solutions were published in 1958†. In this note I will give not only a solution of this problem but also suggest some methods which lead to integrals of the more general type $\int_0^{\pi} (\log \sin \phi)^n d\phi$ and point out that there are relations between these integrals and some Dirichlet series.

2. The starting point is the function

$$D(s) = \sum_{n=0}^{\infty} \frac{1\cdot 3 \cdot \cdots \cdot (2n-1)}{2\cdot 4 \cdot \cdots \cdot 2n} \cdot \frac{1}{(2n+1)^s} \qquad (s = \sigma + \tau i). \tag{1}$$

The right-hand series is a Dirichlet series which converges absolutely for $\sigma > \frac{1}{2}$. From $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$ follows that

$$\frac{1}{(2n+1)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-(2n-1)x} x^{s-1} dx. \tag{2}$$

Considering, on the other hand, the uniform convergence of the series

$$\sum_{n=0}^{\infty} \frac{1\cdot 3 \cdot \cdots \cdot (2n-1)}{2\cdot 4 \cdot \cdots \cdot 2n} (e^{-x})^{2n+1} \qquad \text{for } x > 0,$$

we easily derive that

$$D(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{\sqrt{(e^{2x} - 1)}}\, dx \qquad \text{for } \sigma > \tfrac{1}{2}. \tag{3}$$

The substitutions $e^{-x} = \sin \phi$ and $s = n$ (where n is an integer) give that

$$D(n) = \frac{(-1)^{n-1}}{(n-1)!} \int_0^{\frac{1}{2}\pi} (\log \sin \phi)^{n-1} d\phi. \tag{4}$$

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* Nieuw Arch. Wisk. (ser. 3), vol. 5, 1957, p. 48.

† Wisk. Opgaven, vol. 20, 1958, pp. 9–10.

As the series

$$\sum_{n=0}^{\infty} \frac{(\log \sin t)^n}{n!} x^n = (\sin t)^x$$

converges uniformly for each $0<x<\frac{1}{2}\pi$, a term-by-term integration may be applied, which gives

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \int_0^{\frac{1}{2}\pi} (\log \sin t)^n dt = \sum_{n=0}^{\infty} (-1)^n D(n+1)x^n = \int_0^{\frac{1}{2}\pi} (\sin t)^x dt = \tfrac{1}{2}\sqrt{\pi}\, \frac{\Gamma(\frac{1}{2}x+\frac{1}{2})}{\Gamma(\frac{1}{2}x+1)}, \quad |x|<1. \tag{5}$$

In a similar way we find that

$$\sum_{n=0}^{\infty} D(n+1)x^n = \sqrt{\pi}\, \frac{1}{x}\, tg\, \tfrac{1}{2}x\pi\, \frac{\Gamma(\frac{1}{2}x+1)}{\Gamma(\frac{1}{2}x+\frac{1}{2})}. \tag{6}$$

By multiplication of (5) and (6) we find that

$$\sum_{n=0}^{\infty} (-1)^n D(n+1)x^n \sum_{n=0}^{\infty} D(n+1)x^n = \sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} B_n \pi^{2n} x^{2n-2}, \tag{7}$$

where the Bernoulli numbers B_n are defined by

$$\frac{1}{x}\, tg\, \tfrac{1}{2}x\pi = 2\sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} B_n \pi^{2n-1} x^{2n-2}.$$

By means of comparison of the coefficients of x^{2n-2} in both members of (7) we find that

$$D(1)D(2n-1) - D(2)D(2n-2) + \cdots + D(2n-1)D(1) = \frac{2^{2n}-1}{(2n)!}\pi^{2n}B_n, \tag{8}$$

which is a recurrence relation for the D's. From Dirichlet's series follows:

$$D(1) = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{1}{2n+1} = \arcsin 1 = \tfrac{1}{2}\pi,$$

and from (3)

$$D(2) = -\int_0^{\frac{1}{2}\pi} \log \sin t\, dt = \tfrac{1}{2}\pi \log 2.$$

From (8) we find for $n=2$ that $D(1)D(3)-D(2)D(2)+D(3)D(1)=(1/48)\pi^4$; so that

$$D(3) = \frac{1}{48}\pi^3 + \tfrac{1}{4}\pi(\log 2)^2 = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\log \sin x)^2 dx,$$

which is Bremekamp's formula.

3. Another recurrence relation can be derived from (5) by differentiation; the result is:

$$(9)\quad \sum_{n=1}^{\infty}(-1)^n nD(n+1)x^{n-1} = \tfrac{1}{2}\left[\psi\left(\tfrac{1}{2}x+\tfrac{1}{2}\right)-\psi\left(\tfrac{1}{2}x+1\right)\right]\sum_{n=0}^{\infty}(-1)^n D(n+1)x^n,$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$.

We now give the following formula:

$$(10)\quad \psi\left(\tfrac{1}{2}x+\tfrac{1}{2}\right)-\psi\left(\tfrac{1}{2}x+1\right) = -\log 4 + 2\sum_{n=1}^{\infty}(-1)^{n+1}\left(1-\frac{1}{2^n}\right)\zeta(n+1)x^n,$$

where $\zeta(n+1)=\sum_{k=1}^{\infty}k^{-n-1}$ denotes Riemann's zeta function. The formula (10) can be derived by expanding the function $f(x)=\psi(\frac{1}{2}x+\frac{1}{2})-\psi(\frac{1}{2}x+1)$ in a series, with the help of Maclaurin's formula, and with the following result, known from the theory of gamma functions,

$$\psi(x) = \int_0^{\infty}\left(e^{-a}-\frac{ae^{-ax}}{1-e^{-a}}\right)\frac{1}{a}\,da.$$

The substitution of (10) in (9) gives

$$\sum_{n=1}^{\infty}(-1)^n nD(n+1)x^{n-1} = \left[-\log 2+\sum_{n=1}^{\infty}(-1)^{n+1}\left(1-\frac{1}{2^n}\right)\zeta(n+1)x^n\right]\cdot\sum_{n=0}^{\infty}(-1)^n D(n+1)x^n,$$

from which, through comparison of the coefficients of x^{n-2}, it follows that

$$(11)\quad (n-1)D(n) = D(n-1)\cdot\log 2+\sum_{k=1}^{n-2}\left(1-\frac{1}{2^k}\right)\zeta(k+1)D(n-k-1).$$

From the theory of zeta functions we know that

$$\zeta(2m) = \frac{(2\pi)^{2m}}{4m(2m-1)!}B_m.$$

It is not possible to continue to express all D's, one after the other, in terms of "known" constants since no such expression is known for $\zeta(2m+1)$; however, all D's can be expressed in terms of the zeta function of positive integers. Thus

$$D(4) = \frac{1}{12}\pi(\log 2)^3+\frac{1}{48}\pi^3\cdot\log 2+\frac{1}{8}\pi\cdot\zeta(3),$$

$$D(5) = \frac{19}{11520}\pi^5+\frac{1}{96}\pi^3(\log 2)^2+\frac{1}{48}\pi(\log 2)^4+\frac{1}{8}\pi\cdot\log 2\cdot\zeta(3),\quad \text{etc.}$$

REMARKS ON THE FIBONACCI SERIES MODULO m

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In a recent paper D. D. Wall [1] was concerned with determining the length of the period of the recurring series obtained by reducing a Fibonacci series modulo m. In this paper we will use the same notation as in [1].

Thus, u_n denotes the Fibonacci series with $u_0=0$ and $u_1=1$, where $u_{n+1}=u_n+u_{n-1}$. Also, $k(m)$ denotes the length of the period of u_n mod m and as in [1] we let $m=p^e$, where p is a prime number. In [1] Wall poses a question that has so far remained unanswered: "The most perplexing problem we have met in this study concerns the hypothesis $k(p^2)\neq k(p)$. We have run a test on a digital computer which shows that $k(p^2)\neq k(p)$ for all p up to 10,000; however, we cannot yet prove that $k(p^2)=k(p)$ is impossible." This paper furnishes a proof of the hypothesis $k(p^2)\neq k(p)$ under certain mild conditions.

THEOREM 1. *If c and p are relatively prime and cp occurs in u_n, then $k(p^2)\neq k(p)$.*

Proof. Let $u_j=cp$ and consider the sequences

(1) u_n, mod p which we will denote by ${}_1u_n$;

(2) u_n, mod p^2 which we will denote by ${}_2u_n$

which begin, respectively, 0, 1, 1, 2, $\cdots$, ${}_1u_{j-1}=Rp+Q$, ${}_1u_j=cp\equiv 0$, ${}_1u_{j+1}\equiv Q$, $\cdots$ (mod p) and 0, 1, 1, 2, $\cdots$, ${}_2u_{j-1}=Rp+Q$, ${}_2u_j=cp\equiv cp$, ${}_2u_{j+1}\equiv(c+R)p+Q$, $\cdots$ (mod p^2), where $0<Q<p$. It can be shown by mathematical induction that

(3) $${}_1u_{tj-1}\equiv Q^t, \qquad {}_1u_{tj}\equiv 0 \pmod{p};$$

(4) $${}_2u_{tj-1}\equiv tRpQ^{t-1}+Q^t, \qquad {}_2u_{tj}\equiv ctpQ^{t-1} \pmod{p^2}.$$

To see that (4) holds, we note that for $t=1$, the formulas hold. Next, assume the formulas hold for $t\leq i$, then ${}_2u_{ij-1}\equiv iRpQ^{i-1}+Q^i$ and ${}_2u_{ij}\equiv cipQ^{i-1}$ (mod p^2). Now consider the new sequence U_n with $U_0=iRpQ^{i-1}+Q^i\equiv{}_2u_{ij-1}$ and $U_1=cipQ^{i-1}\equiv{}_2u_{ij}$ (mod p^2). But, by the well-known formula for a Fibonacci series in [1], $f_n=u_nb+u_{n-1}a$, where $f_1=a$, $f_2=b$ and $f_{n+1}=f_n+f_{n-1}$, we have $U_j=u_j(cipQ^{i-1})+u_{j-1}(iRpQ^{i-1}+Q^i)$ or $U_j\equiv(i+1)RpQ^i+Q^{i+1}\equiv{}_2u_{(i+1)j-1}$ (mod p^2), and $U_{j+1}=u_{j+1}(cipQ^{i-1})+u_j(iRpQ^{i-1}+Q^i)$ or $U_{j+1}\equiv(i+1)cpQ^i\equiv{}_2u_{(i+1)j}$ (mod p^2). Hence (4) holds; and (3) is implied by (4).

We will, therefore, obtain in the series (1) $\cdots$, ${}_1u_{tj-1}\equiv Q^t$, ${}_1u_{tj}\equiv 0$ (mod p), and in (2) $\cdots$, ${}_2u_{tj-1}\equiv tRpQ^{t-1}+Q^t$, ${}_2u_{tj}\equiv ctpQ^{t-1}$ (mod p^2).

Now series (1) will first repeat when Q belongs to t mod p. (In other words t is the smallest number satisfying $Q^t\equiv 1$ mod p.) In this case, p^2 does not divide ${}_2u_{tj}\equiv ctpQ^{t-1}$ since t divides $p-1$, which means series (2) does not repeat with sequence (1). This proves our theorem.

THEOREM 2. *Let c and p be relatively prime, $e \leqq d$, and $u_j = cp^d$ be the first multiple of p to occur in u_n. Then $k(p^e) = k(p)$ if and only if u_{j-1} has the same order* mod p *and* mod p^e.

Proof. Since u_j is the first multiple of p to occur in u_n, the period of u_n mod p will be jt where u_{j-1} belongs to t mod p. But u_j is also the first multiple of p^e to occur in u_n and so its period is equal to js where u_{j-1} belongs to s mod p^e. Therefore if $k(p) = k(p^e)$ we have $k(p) = jt = js = k(p^e)$ which implies $t = s$. Conversely, under the same hypotheses, if u_{j-1} has the same order mod p and mod p^e it is obvious that $k(p^e) = k(p)$.

Remarks. In [2] Kraitchik has a table of u_n in their prime factorization for odd n up to $n = 129$, with a few missing entries, and none of the u_n listed satisfy the hypothesis of Theorem 2 for $1 < e \leqq d$ in this paper. Furthermore, I have computed all of the u_n up to $n = 50$ using Lehmer's prime and factor tables plus the tables in [2] as a check and, again, none of the u_n satisfy the hypothesis of our Theorem 2 for $1 < e \leqq d$. Could it be that the mild conditions of Theorem 1 are strong enough to apply to all prime numbers? That is to say, one would like to make Theorem 1 read: *If c and p are relatively prime, then cp occurs in u_n and $k(p^2) \neq k(p)$.*

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2. M. Kraitchik, Recherches sur la Théorie des Nombres, vol. 1, Paris, 1924, pp. 77–80.

NOTE ON THE DISTRIBUTIVE LAWS (Supplement)

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J. L. Kelley [1] defines a ring to be a system which we called a c-ring in [2]. Then he defines a field to be a system which is a ring (in his sense) such that the set of all nonzero elements forms a multiplicative commutative group. In this supplementary note, we show that the definition of a field in Kelley's sense coincides with that in the ordinary sense.

We use terminologies and results in [2] freely.

We define a *w-division ring* (*c-division ring*) to be a w-ring (c-ring) F such that F has at least two elements and $F - \{0\}$ forms a multiplicative group. When the multiplicative group is commutative, it is called a *w-field* (*c-field*). Hence a field in Kelley's sense is a c-field in our definition.

THEOREM 1. *If a w-division ring F contains an element which is neither zero nor the defining element, then F is a division ring (in the ordinary sense).*

Proof. It suffices to show that F is a ring. By [2] Lemma 3, we have, in F, $e^2 = e$, where e is the defining element of F. If e were not 0, e would be an idempotent element of the multiplicative group $F^* = F - \{0\}$, and so e would be the identity element of F^*. Then for any $x \in F^*$, we have, using [2] Lemma 3 again,

$x=xe=e$. Hence F^* would consist of solely the defining element e, contradicting the assumption. Hence $e=0$, and therefore F is a ring.

THEOREM 2. *Every c-division ring is a division ring. Conversely, every division ring is a c-division ring.*

Proof. Let F be a c-division ring. By [2] Theorem 3, F can be considered as a w-division ring of order 3 or 1 with the defining element $e=00$. If F were of order 3, the element $2e$ would be neither zero nor the element e. Hence by Theorem 1, F would be a division ring and so $e=0$, which is absurd. Hence F is of order 1 and therefore is a division ring. The converse part of the theorem is evident.

COROLLARY. *Every c-field is a field and conversely every field is a c-field.*

References

1. J. L. Kelley, General Topology, New York, 1955.
2. T. Saitô, Note on the distributive laws, this MONTHLY, vol. 66, 1959, pp. 280–283

CORRECTION

In the note *On some equations involving functions* $\phi(n)$ *and* $\sigma(n)$ by A. Makowski (this MONTHLY, vol. 67, 1960, pp. 668–670), the first number in (b) on page 670 should be $2^2\cdot 3\cdot 5\cdot 13$.

CLASSROOM NOTES

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SO-CALLED "EXCEPTIONAL" EXTREMUM PROBLEMS

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Although the invitation "rejoinders to earlier notes are encouraged" is no longer printed in the MONTHLY, I should like to make a rejoinder to Ogilvy's Classroom Note [1]. I shall refer also to Thurston [2], Oakley [3], and Walsh [4].

Ogilvy maintains that the "elementary calculus method" of finding maxima and minima may fail for two reasons, and he gives an example of each. The second failure he imputes to an "unsuitable independent variable" and remarks: "what we should like is a criterion for judging in advance the suitability of an independent variable. Unfortunately this seems rather too much to ask."

I maintain that the elementary method does not fail in either case and that therefore no criterion is needed. By the elementary method I mean the method

suggested by common sense: we examine the stationary values of the function being investigated, the values where the derivative fails to exist, and the endpoint values. The greatest of these is the maximum and the least the minimum, provided of course the function does have a maximum and a minimum. A similar method occurs in Walsh's paper (the case where there is just one extremum in the range under consideration) and is sufficient to exemplify the principle that we should realize precisely what we are looking for, rather than mechanically apply a learnt test.

Ogilvy's first problem is essentially to maximize $(50-x)(100+x)$ under the conditions $0<x<50$. He writes that if we proceed "in the usual fashion" then "x comes out to be -25, which is not permitted by the conditions of the problem." But to get this result, the standard (and, I would have hoped, usual) method must be replaced by something like "set the derivative equal to zero and solve."

The second problem is essentially:—

$$\text{Minimize } (x-1)^2+y^2 \text{ under the condition } y^2=4x.$$

This is clearly equivalent to: Minimize $(x-1)^2+4x$ under the condition $x\geqq 0$. Because the derivative is always positive throughout the range considered, the minimum occurs when $x=0$. Ogilvy states that his method of solution "yields $x=-1$, a point not on the parabola."

In both cases, the incorrect results can be imputed to *neglect of the domain of the function under investigation.* (*E.g.*, the minimum of $(x-1)^2+4x$ with no restriction on x does occur when $x=-1$; the maximum of $(50-x)\cdot(100+x)$ does occur when $x=-25$.) In my note, I said regretfully that the concept of domain is apt to be pushed into the background, and gave an example (from a well-known textbook) of a problem whose solution goes wrong if domains are neglected. There are no such things as "unsuitable" variables: the use of this term is an attempt to put on the variable the blame which rightly belongs to an illogical method of solution.

Why are both common sense and Walsh's thirteen-year-old paper neglected? It must be because teaching is too mechanical: finding stationary points is a convenient way for finding certain local maxima and minima; and the *technique* of setting the derivative equal to zero is emphasized at the expense of *understanding* what the problem is and why the technique is being used. Why else would a student expect to find the least value of $a-ex$ as x varies from $-a$ to a by differentiation? And yet Oakley found that "even an able student" might do this.

The fact is that as one goes through life most maximum and minimum problems are endpoint problems. How fast do I run to get to the telephone as quickly as possible? Answer: as fast as I can—an endpoint solution. Of course, most of these problems are trivial; the interesting ones are where there is some balancing factor: in maximizing $x(1-x)$ the increase in x balances the decrease in $(1-x)$.

However, we teachers of calculus seem to have so lost our sense of proportion that we regard these as normal and the endpoint solutions as wicked exceptions.

To show how far our sense of proportion has been lost, let us consider the following problem. Choose P between P_0 and the vertex P_m on the parabola $y=9x^2-28x+24$ (Fig. 1), complete the shaded region and rotate it about the y-axis. Where should P be taken to give a maximum volume? Because P is restricted to a finite closed interval, the endpoint maximum (at P_m) and minimum (at P_0) stare us in the face; it would certainly be possible to have extrema in between, but at any rate the endpoint values are the most obvious suspects.

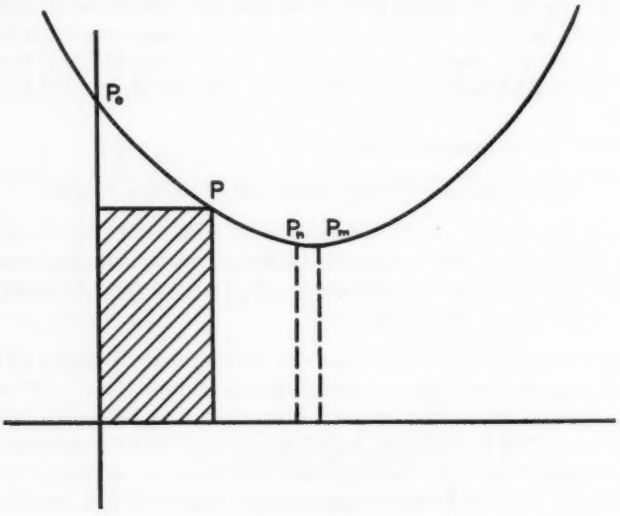

FIG. 1

Indeed, no one would doubt that the value given by P_m is greater than that given by P_n. And yet the problem is described [5] as "A *well-concealed* endpoint maximum"! (My italics.) However, we are not alone in making this mistake. I heard that, until recently, geographers, when asked for the highest point in Florida, would cite the highest of Florida's few hills. Now, however, someone has noticed that the highest point in Florida is on the Alabama border: an endpoint maximum.

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THE AREA OF THE ELLIPSE

ROGER BURR KIRCHNER, Harvard University

Consider the ellipse E whose equation is $x^2/a^2+y^2/b^2=1$, where $a, b>0$. Suppose that we make the transformation $(x, y)\rightarrow(ax, by)$. The x- and y-axes are stretched (or compressed) into new axes which we call the x' and y' axes. If the x-y coordinates of a point P are (x, y) and its x'-y' coordinates are (x', y'), then clearly

$$x = ax', \qquad y = by'. \tag{1}$$

Thus, in the x'-y' system, E has the equation $x'^2+y'^2=1$.

Let Δ be a triangle with one side parallel to the x-axis. Suppose that the x-y coordinates of its vertices are (x_1, y_1), (x_2, y_1), and (x_3, y_3), and the corresponding x'-y' coordinates are (x_1', y_1'), (x_2', y_1'), and (x_3', y_3'). If the area of Δ in the x-y system is denoted by A, and in the x'-y' system by A', then

$$A = \tfrac{1}{2}\,|\,x_2 - x_1\,|\cdot|\,y_3 - y_1\,|\,, \qquad A' = \tfrac{1}{2}\,|\,x_2' - x_1'\,|\cdot|\,y_3' - y_1'\,|. \tag{2}$$

Hence, from the equations (1), we find that $A=abA'$.

Since any region bounded by a polygonal curve can be broken up into triangles of the type Δ, we see that its area in the x-y system is just ab times its area in the x'-y' system. The same must then be true for regions whose boundaries are limits of polygonal curves. Such a region is the ellipse E. Since E is just the unit circle in the x'-y' system, its x'-y' area is π. Hence the area of the ellipse with equation $x^2/a^2+y^2/b^2=1$ is πab.

SIMPLE CONSTRUCTIONS OF NONDIFFERENTIABLE FUNCTIONS AND SPACE-FILLING CURVES

WILLIAM C. SWIFT, Rutgers, The State University

Although the beginning calculus student is commonly informed of the existence of continuous functions which are nowhere differentiable, the demonstration of the celebrated examples of such functions is beyond his competence. We exhibit here a function of this class which should be meaningful to anyone comprehending the binary (base 2) representation of real numbers. In a similar manner we define functions $u(t)$ and $v(t)$, each continuous and nowhere-differentiable, with the added property that the image of $\{t\,|\,0\leqq t\leqq 1\}$ completely fills the square $\{(u, v)\,|\,0\leqq u\leqq 1,\ 0\leqq v\leqq 1\}$.

1. The value of the function $f(x)$ for x in the interval $[0, 1]$ is based on the ternary (base 3) decimal expansion of x. Let

$$x = .x_1 \cdots x_n \cdots \quad \text{(base 3)},$$

in which of course $x_n=0$, 1 or 2. Then $f(x)$ is given by the binary decimal

$$f(x) = .y_1 \cdots y_n \cdots \quad \text{(base 2)},$$

where $y_n = 0$ or 1 as determined by the recursive rule: $y_1 = 1$ if and only if $x_1 = 1$; and $y_{n+1} = y_n$ if and only if $x_{n+1} = x_n$.

Example. Let $x = .2012011222 \cdots = .2012012000 \cdots$ (base 3). Then

$$f(x) = .0101011000 \cdots = .0101010111 \cdots \quad \text{(base 2)}.$$

The reader may easily verify that for every x expressible as two distinct ternary decimals, the two binary decimal expansions for $f(x)$ define the same real number.

The continuity of $f(x)$ for each x follows immediately from the observation that $f(x)$ and $f(x')$ have binary decimal expansions agreeing in the first n digits whenever x' has a ternary expansion agreeing in the first n digits with the ternary expansion of x or with either of the two ternary expansions of x, as the case may be.

To establish that $f(x)$ is nondifferentiable, suppose

$$x = .x_1 \cdots x_n x_{n+1} x_{n+2} \cdots x_{n+k} \cdots \quad \text{(base 3)}$$

with

$$f(x) = .y_1 \cdots y_n y_{n+1} y_{n+2} \cdots y_{n+k} \cdots \quad \text{(base 2)}.$$

Clearly for any given n we can select $x'_{n+1}, x'_{n+2}, \cdots, x'_{n+k}, \cdots$ such that, for

$$x' = .x_1 \cdots x_n x'_{n+1} x'_{n+2} \cdots x'_{n+k} \cdots \quad \text{(base 3)},$$

we have

$$f(x') = .y_1 \cdots y_n y'_{n+1} y_{n+2} \cdots y_{n+k} \cdots \quad \text{(base 2)},$$

where $y'_{n+1} \neq y_{n+1}$ but otherwise corresponding digits of $f(x)$ and $f(x')$ are identical. Under these circumstances

$$0 < |x' - x| \leqq 3^{-n} \quad \text{and} \quad |f(x') - f(x)| = 2^{-(n+1)}.$$

Therefore

$$\left| \frac{f(x') - f(x)}{x' - x} \right| \geqq \tfrac{1}{2}\left(\tfrac{3}{2}\right)^n.$$

It clearly follows that $f(x)$ is nowhere differentiable.

2. In a variation on the manner of definition of $f(x)$ we now define functions $u(t)$ and $v(t)$ for t in the interval $[0, 1]$. Let t be expressed in the notation of a decimal of base 6:

$$t = .t_1 \cdots t_n \cdots \quad \text{(base 6)},$$

where of course $t_n = 0, 1, 2, 3, 4$ or 5. Corresponding to this value of t, $u(t)$ and $v(t)$ are given by the binary decimals

$$u(t) = .u_1 \cdots u_n \cdots \quad \text{(base 2)},$$

$$v(t) = .v_1 \cdots v_n \cdots \quad \text{(base 2)},$$

where $u_n=0$ or 1 and $v_n=0$ or 1 as given by the following rule:

First, $u_1=0$ if and only if $t_1=0$, 1 or 2, and $v_1=0$ if and only if $t_1=0$, 1 or 3.

To determine u_{n+1} and v_{n+1} we distinguish six cases:

1. if $t_{n+1}=0$ or 5 and $t_{n+1}=t_n$, then $u_{n+1}=u_n$ and $v_{n+1}=v_n$;
2. if $t_{n+1}=0$ or 5 and $t_{n+1}\neq t_n$, then $u_{n+1}\neq u_n$ and $v_{n+1}\neq v_n$;
3. if $t_{n+1}=1$, then $u_{n+1}=0$ and $v_{n+1}=0$;
4. if $t_{n+1}=2$, then $u_{n+1}=0$ and $v_{n+1}=1$;
5. if $t_{n+1}=3$, then $u_{n+1}=1$ and $v_{n+1}=0$;
6. if $t_{n+1}=4$, then $u_{n+1}=1$ and $v_{n+1}=1$.

That $u(t)$ and $v(t)$ are continuous functions which are nowhere differentiable may be demonstrated following the procedure accorded $f(x)$ above. Moreover, considering the functions simultaneously, it is immediate from their definition that every point of the square $\{(u, v)\,|\,0\leqq u\leqq 1, 0\leqq v\leqq 1\}$ is the image of some t in the interval $[0, 1]$.

ON THE USE OF TABLES TO OBTAIN CONFIDENCE INTERVALS

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In some statistical problems it may appear that one needs tables that are not readily available. Sometimes tables possessed by nearly all statisticians are better adapted to the situation. Two examples of this follow.

When the general method of obtaining confidence intervals is applied to the binomial case, it leads to the equations

$$\sum_{y=k}^{n}\binom{n}{y}p^y(1-p)^{n-y}=\tfrac{1}{2}\alpha, \tag{1}$$

$$\sum_{y=0}^{k}\binom{n}{y}p^y(1-p)^{n-y}=\tfrac{1}{2}\alpha, \tag{2}$$

where n, k, and α are all known numbers. If the solution of (1) is p_1 and that of (2) is p_2, then (p_1, p_2) is a confidence interval for p with coefficient $1-\alpha$. That is, $\Pr[p_1<p<p_2]=1-\alpha$.

The above equations can be solved with the aid of binomial tables. However, frequently tables are not readily available and, even if they are, some kind of interpolation is necessary. It is well known that

$$\frac{n!}{(k-1)!(n-k)!}\int_0^p w^{k-1}(1-w)^{n-k}dw=\sum_{y=k}^{n}\binom{n}{y}p^y(1-p)^{n-y}.$$

This is easily demonstrated by repeated parts integration. Thus tables of the incomplete Beta function could be used to solve (1). The difficulties are the same as with the binomial tables.

Good F tables are generally available and adapted to this kind of problem. Letting

$$w = \frac{\dfrac{kF}{n-k+1}}{1+\dfrac{kF}{n-k+1}},$$

it is found that

$$\Pr[0 < w < p] = \Pr\left[0 < F < \frac{n-k+1}{p}\,\frac{p}{1-p}\right],$$

where F has $2k$ and $2(n-k+1)$ degrees of freedom. Thus to solve (1), solve instead

$$\Pr\left[0 < F < \frac{n-k+1}{k}\,\frac{p}{1-p}\right] = \tfrac{1}{2}\alpha.$$

When $\{(n-k+1)/k\}\{p/(1-p)\}$ is equated to the tabulated F value one has a linear equation in p to solve for p_1. Similarly (2) may be replaced by

$$\Pr\left[F > \frac{n-k}{k+1}\,\frac{p}{1-p}\right] = \tfrac{1}{2}\alpha,$$

where F has $2(k+1)$ and $2(n-k)$ degrees of freedom and the solution is p_2.

In the Poisson case where

$$f(x;m) = e^{-m}\frac{m^x}{x!}, \qquad x = 0, 1, \cdots,$$

one is led to the equations

$$\sum_{y=y'}^{\infty} e^{-nm}\frac{(nm)^y}{y!} = \tfrac{1}{2}\alpha, \tag{3}$$

$$\sum_{y=0}^{y'} e^{-nm}\frac{(nm)^y}{y!} = \tfrac{1}{2}\alpha, \tag{4}$$

where n, α and $y' = \sum_{i=1}^{n} x_i$ are known.

The solutions of (3) and (4) yield m_1 and m_2 such that

$$\Pr[m_1 < m < m_2] = 1 - \alpha.$$

Repeated parts integration gives

$$\int_m^{\infty} \frac{1}{\Gamma(n+1)}\,x^n e^{-x}dx = \sum_{y=0}^{n} e^{-m}\frac{m^y}{y!},$$

but both the Poisson tables and the incomplete Gamma tables possess the same objections mentioned previously. Since χ^2/degrees of freedom is extensively

tabulated and readily available (see Dixon and Massey) let $x=\frac{1}{2}w$. Then w is χ^2 with $2n+2$ degrees of freedom so that

$$\sum_{\nu=0}^{n} e^{-m}\frac{m^\nu}{\nu!} = \Pr[w > 2m] = \Pr\left[\frac{w}{2n+2} > \frac{m}{n+1}\right],$$

$$\sum_{\nu=0}^{\nu'} e^{-nm}\frac{(nm)^\nu}{\nu!} = \Pr\left[z > \frac{nm}{\nu'+1}\right],$$

where z is χ^2 divided by degrees of freedom $2\nu'+2$. If the tabular value is b, then $m_2=b(\nu'+1)/n$. Similarly

$$\sum_{\nu=\nu'}^{\infty} e^{-nm}\frac{(nm)^\nu}{\nu!} = \Pr\left[z < \frac{nm}{\nu'}\right] = \tfrac{1}{2}\alpha$$

gives $m_1=a\nu'/n$ where z is χ^2 divided by degrees of freedom $2\nu'$ and a is the tabulated value.

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A CHARACTERIZATION OF COMPACT METRIC SPACES

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1. Introduction. It is well known that a compact metric space is bounded; the converse of course is not true, for example, $M: 0<x<1$ with the usual metric. It is possible to remetrize M so that M becomes unbounded relative to the new metric. We shall see in the next section that every noncompact metric space can be remetrized so that it becomes unbounded relative to the new metric.

2. A characterization of compact metric spaces.

LEMMA. *If M is a metric space, then M is compact if and only if every $f: M\to R$ is bounded where f is continuous and R is the set of all real numbers with the usual topology.*

A proof of this lemma is given in [1], page 119.

THEOREM. *Let M be a metric space with metric d. Then M is compact if and only if M is bounded relative to d^* where d^* is any metric for M equivalent to d.*

Proof. The necessity is well known. To show the sufficiency let M be bounded relative to d^* where d^* is any metric equivalent to d. Assert then that M is compact. Deny. Then by the above lemma there exists an $f^*: M\to R$ such that f^* is continuous and unbounded, R being the space of reals. We now define a new metric for M as follows: for x, y in M let $d^\sharp(x, y)=d(x, y)+|f^*(x)-f^*(y)|$. The theorem is proved when we show that (a) $d^\sharp$ is a metric for M equivalent to

d and (b) M is unbounded relative to $d^\sharp$. To show (a), we note that (1) $d^\sharp(x, y) \geqq 0$, (2) $d^\sharp(x, y) = 0$ if and only if $x = y$, (3) $d^\sharp(x, y) = d^\sharp(y, x)$, (4) $d^\sharp(x, z) = d(x, z) + |f^*(x) - f^*(z)| \leqq d(x, y) + d(y, z) + |f^*(x) - f^*(y)| + |f^*(y) - f^*(z)| = d^\sharp(x, y) + d^\sharp(y, z)$, and finally (5) let $y \in M$ and $y_i \in M$ for $i = 1, 2, \cdots$ and suppose $\lim d^\sharp(y, y_i) = 0$. Then $\lim d(y, y_i) = 0$ since $d(y, y_i) \leqq d^\sharp(y, y_i)$. Conversely let $\lim d(y, y_i) = 0$. Then $\lim f^*(y_i) = f^*(y)$ since f^* is continuous and thus $\lim d^\sharp(y, y_i) = \lim d(y, y_i) + \lim |f^*(y) - f^*(y_i)| = 0$. To show (b) choose x_i in M so that $\lim |f^*(x_i)| = \infty$. Then

$$\begin{aligned} d^\sharp(x_i, x_1) &= d(x_i, x_1) + |f^*(x_i) - f^*(x_1)| \\ &\geqq |f^*(x_i) - f^*(x_1)| \\ &\geqq ||f^*(x_i)| - |f^*(x_1)|| . \end{aligned}$$

Thus $d^\sharp(x_i, x_1) \to \infty$.

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MATHEMATICAL EDUCATION NOTES

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THE TEACHING OF MATHEMATICS*

Professor J. P. Mathieu of the French delegation presented an introduction to the teaching of mathematics to physicists:

I have tried to find out the opinion of my French colleagues, but unfortunately without success, and I must therefore give you a personal view rather than a national one.

First we must decide on the level of education we are concerned with. In secondary education a number of important problems arise in the teaching of mathematics. At this level education is intended to give a general culture, and the teaching of mathematics can be considered as being an end in itself, giving the pupils a certain attitude of mind and a way of reasoning of which a mathematician is very proud and jealous.

* This part of Chapter II of the Proceedings of the International Conference on Physics Education, and additional excerpts from this chapter to be published next month, are used with the permission of the publishers, The Technology Press, Massachusetts Institute of Technology, and John Wiley and Sons, Inc. New York and London. The Conference was held July 18 to August 4, 1960, in UNESCO House, Paris. The proceedings were edited by Sanborn C. Brown and Norman Clarke.

Problems do arise, however, at this level between the teaching of mathematics and physics—problems of syllabus and problems of teaching method. In France from a very early stage we give importance to abstract education, more, probably, than in many other countries, and this raises the question of the relation between mathematics and physics.

At the university level the student has made his choice and is going to specialize. He knows the road he is taking, and what he wants from mathematics is an instrument, rather than a formative discipline. One, of course, does not exclude the other, and the statement in the report of the (British) Institute of Physics* is quite correct when it says that if an instrument must be used, let it be used with intelligence and not mechanically.

A special problem is the mathematics that is essential in the study of physics by biologists and medical students, for whom physics is an accessory discipline.

I will not, however, dwell on this question, as what interests us above all is the training of the physicists. I will not treat the training of the chemist as a separate problem, as Professor Aigrain did. In my opinion, the same mathematical education can be given to the physicist and the chemist. This does not happen in my country, but I do not consider it wrong that it does happen in other countries.

Here again, one must be careful to stipulate at what level the teaching of mathematics is to be. As the British report I have quoted so rightly states, it is the degree of emphasis on generalization or rigor that must be decided in teaching of mathematics for the earlier ages.

People often suppose that there is a difference between the spirit in which mathematics should be taught and the spirit that should animate the teaching of physics, but if one looks at the theoretical form which the concepts of physics take when they have matured, there is no difference of spirit. There is in fact no difference of spirit in the scientific disciplines generally.

What then do we consider the minimum of mathematics that must be taught to physicists, and what need not be taught? This depends on the level considered and is a question of utility.

In France, the mathematician wishes to teach very seriously in order to educate the pupil in a general way. What are the advantages of so doing? First, economy of thought. If we create concepts that have a very wide general applicability, then we teach at one and the same time a number of ideas that can appear to be unconnected. The physicst knows this well. What are the disadvantages of this tendency to discuss ideas in their most general form?

In the first place, it is illusory to abstract oneself from reality. An axiomatic system does not constitute a science, and this may be missed by a student to whom only general notions are taught.

Further, it will take much more time and effort to teach mathematics in

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* The Teaching of Mathematics to Physicists. The Institute of Physics and The Physical Society, London, 1960.

this way than in a more practical manner.

Let us look at the situation in practice.

The physicist will require from the mathematician a suitable syllabus, and, as part of this suggested syllabus will not be of great use to him, it will be deleted. The Institute of Physics' report, in fact, gives a syllabus of this type which I find interesting.

Second, the physicist will expect from the mathematician a method, and this point has been very much discussed in France. The mathematician needs more generalized proofs than we physicists do. In other words, the physicist is ready to believe the mathematician when he gives him details of existence theorems. He will believe, because he has every reason to believe.

There is a third point. It is of the greatest importance to the physicist to be able to interpret the solutions of equations, and on this the mathematician—at least in our country—gives no help to the physicist.

What is a practical way out of these difficulties? I believe that there must be close collaboration between the teacher of mathematics and the physicist who is an expert in mathematics. In France the situation is very unsatisfactory. We never seem able to arrange for a physicist expert in mathematics to teach mathematics to future physicists.

We must, of course, remember that, although the mathematician uses his physical concepts drawn from reality, it is also true that certain abstract and disinterested mathematical studies have been found eventually to be of use to physicists. We must, therefore, leave mathematicians to make their own researches.

The important thing is that the mathematics which is taught should be of some use.

The discussion after Professor Mathieu's address followed two main lines: whether or not mathematics for physicists should be taught by mathematicians or physicists, and detailed discussions on what subject matter of mathematics should be included, and in what form.

On the first question, as to whether or not mathematics for physicists should be taught by mathematicians or not, no agreement was achieved, and strong opinions were stated on both sides.

During the discussion of techniques and the content of a mathematics course, several interesting comments were made by Professor A-K. M. Kinani of the United Arab Republic:

We have been impressed that in courses of mathematics for physicists, and also in the books on mathematics especially for physicists, all examples for giving the fundamental concepts of mathematics are, in general, abstract and dull. We have tried giving physical examples to the students to learn the applications of mathematics to physics, both at secondary-school and university levels. For example, in secondary school when we study fractions and variables, we use the physical example of the length of a bar as a function of temperature.

As examples of derivatives we use the connection between electric current and the quantity of electric charge.

At the university we take as an example of the exponential function the discharge of a capacitor or the fundamentals of radioactive disintegration. We have tried this system in our country, and we have seen that this method interests the students very much.

Dr. J. Topping of the United Kingdom delegation pointed out the following:

One of the most important developments of mathematics in recent years is the development of numerical mathematics. It is a new and tremendously important branch of mathematics which has meant that problems which could not be tackled before can now be solved. This has led to the use of electronic computers and all sorts of devices of this kind. One of the developments in England in teaching mathematics to physicists in the Diploma in Technology courses is that they are introduced to these machine methods and the general methods of numerical analysis.

At the more advanced level, Professor A. Borsellino of Italy had this to say:

I am a theoretical physicst, and I am very much intersted in the mathematical background for physicists. I think it is one of the greatest dangers in the relationship between physicists and mathematicians at the present time that we do not understand each other on the research plane. Naturally we uncerstand each other on the plane of elementary mathematics, on the plane of mathematics of one century ago. But now we do not understand each other. Essentially mathematicians are now working on subjects that we as physicists know very little about. We should make contact with them at some points to permit real communication from modern mathematics to the physical knowledge of the average physicist, and not let the only contact be through the specialist in the most advanced theoretical physics.

MATHEMATICS AND STATISTICS DEGREES DURING THE DECADE OF THE FIFTIES

CLARENCE B. LINDQUIST, U. S. Office of Education

During the decade of the 1950's the number of doctorates in mathematics and statistics conferred by institutions of higher education in the United States experienced a steady gain, while the number at the master's and bachelor's level decreased during the first half of the decade but rose during the last half, with an especially sharp increase in the late 1950's.

The numbers of degrees that were conferred annually at each level from 1949–50 through 1958–59, together with the percents that these numbers were of all degrees at each level, are shown in Table 1. The data in this table were obtained from the annual series of *Earned Degrees Conferred by Higher Educational Institutions*, published by the U. S. Office of Education. Each annual report covers a period from July 1 of one year to June 30 of the next. No distinction is made

between degrees earned in liberal arts or in preparation for teaching, or in any other category. The criterion that is used in classification is that there be a substantive major in mathematics or statistics. Double majors are counted as one-half in each discipline. Thus, a double major by a student in mathematics and physics is allocated one-half to mathematics and one-half to physics.

TABLE 1

NUMBER OF DEGREES CONFERRED ANNUALLY IN MATHEMATICS AND STATISTICS* BY LEVEL OF DEGREES

Year ending June 30	Bachelor's degrees		Master's degrees		Doctorates (Ph.D., Ed.D., etc.)	
	Number of degrees conferred	Percent† of all degrees at this level	Number of degrees conferred	Percent of all degrees at this level	Number of degrees conferred	Percent of all degrees at this level
1950	6,392	1.47	974	1.67	160	2.41
1951	5,753	1.49	1,109	1.70	184	2.51
1952	4,721	1.42	802	1.26	206	2.68
1953	4,396	1.44	677	1.11	241	2.90
1954	4,090	1.40	706	1.24	227	2.52
1955	4,034	1.40	761	1.31	250	2.83
1956	4,660	1.50	898	1.51	235	2.64
1957	5,546	1.63	965	1.56	249	2.84
1958	6,924	1.89	1,234	1.88	247	2.76
1959	9,019	2.34	1,499	2.16	282	3.01

* Includes actuarial science.

† The percent at the bachelor's level is based upon all bachelor's and first-professional degrees conferred that year.

Facts for the ten-year period. For the ten-year period as a whole, the 55,535 degrees conferred in mathematics and statistics at the bachelor's level constituted 1.62 percent of the total number of bachelor's and first-professional degrees; the 9,625 master's degrees, 1.55 percent of the total number of master's degrees; and 2,281 doctor's degrees, 2.72 percent of the total number of earned doctorates.

Beginning with the 1955–56 survey, the category "mathematics" was replaced by "mathematical subjects," which consisted of two subcategories, "mathematics" and "statistics." Presumably, prior to 1955–56 degrees in statistics were classified by respondents as mathematics. The average number of degrees in statistics at the bachelor's level for the four years, 1955–56 through 1958–59, was 67; at the master's level, 100; and at the doctoral level, 32.

Also beginning with the 1955–56 survey, the Office of Education published the numbers of institutions, by level of degrees, that conferred degrees in the various disciplines for each year. The number of institutions conferring degrees

in mathematics at the bachelor's level rose steadily from 751 in 1955–56 to 871 in 1958–59, and at the master's level from 146 to 177 for the same years. However, at the doctorate level the number of institutions did not change much during any of the four years, averaging about 60 per year. The number of institutions granting bachelor's degrees in statistics averaged about 20 per year; the number conferring master's degrees, about 23 per year; and the number awarding doctorates, about 10 per year. It must be remembered that the number of institutions granting degrees in a specific discipline at a certain level in a given year will usually not represent the universe of institutions authorized to grant these degrees because in some years some of the institutions fail to grant such a degree.

Degrees awarded to women. The percent of mathematics and statistics degrees awarded to women, by level, for the ten-year period, is shown in Table 2, along with the comparable percents for the biological and physical sciences as well as for all fields combined. A fact not shown in the table is that the percent of women receiving bachelor's degrees in mathematics and statistics rose steadily from 22.6 in 1949–50 to 32.7 in 1955–56 and then decreased to 27.9 by 1958–59. The percent for women at the master's level declined from 19.5 in 1949–50 to 16.5 in 1952–53 and then rose to 20.7 by 1958–59. There were no significant trends at the doctorate level.

TABLE 2

THE PERCENT OF DEGREES AWARDED TO WOMEN OVER THE TEN-YEAR PERIOD 1949–50 THROUGH 1958–59

Level	Percent of degrees awarded to women in			
	Mathematics & statistics	Physical sciences	Biological sciences	All disciplines
Bachelor's and first-professional	28.6	12.2	22.2	32.3
Master's	18.8	8.3	20.1	32.0
Doctorate (Ph.D., Ed.D., etc.)	5.3	3.9	10.9	9.8

Junior-year majors in mathematics. In 1957 the Office of Education commenced an annual series of reports, *Junior-Year Science and Mathematics Students*. In the fall of 1957, 9,133 students were reported as junior-year majors in mathematics and statistics. These students would normally be expected to graduate in 1958–59. The number of bachelor's degrees conferred in mathematics and statistics in 1958–59 was 9,019, which gives a ratio of 0.988 for completion of degrees.

The fall 1958 report lists 11,961 junior-year majors in mathematics and

statistics, and the fall 1959 report lists 14,065. If the ratio of the number of graduates in 1958–59 to the number of junior-year majors in the fall of 1957 prevails, we can expect slightly less than 12,000 bachelor degrees in mathematics and statistics in 1959–60, and nearly 14,000 degrees in 1960–61. If these data materialize the number of degrees that will be awarded in 1960–61 will be more than 50 percent greater than the number awarded in 1958–59. It is interesting to note that, based upon similar calculations, the expected increase in the biological sciences for 1960–61 over 1958–59 is expected to be about 7 percent, and in the physical sciences about 1 percent.

Number of doctorates awarded by individual institutions over the ten-year period. Table 3 shows the total number of doctorates in mathematics and statistics awarded by individual institutions for the ten-year period, 1949–50 through 1958–59. The 75 different institutions which awarded the degrees are ranked in decreasing order of the total number each institution awarded during the ten-year period. The first 13 institutions granted more degrees than the remaining 62 did.

The University of California heads the list with 170. It must be remembered, however, that this figure is the total for all campuses, including both Berkeley and Los Angeles. The University of California chooses to report to the Office of Education in this fashion. On the other hand, the University of North Carolina sends separate data to the Office for Chapel Hill, the State College, and the Women's College.

During the 1960's there are certain to be new institutions which have never conferred the doctorate in mathematics and statistics before. Under Title IV of the National Defense Education Act of 1958 alone, fellowships have been awarded, up to the present time, for approved new doctoral programs in mathematics at twelve institutions. These are:

University of Alabama
University of Arizona
Florida State University
University of Idaho
Brandeis University
Montana State College
New Mexico State University
Polytechnic Institute of Brooklyn
Yeshiva University
University of South Carolina
Agricultural and Mechanical College of Texas
Washington State University

Outlook ahead. With the growing recognition of mathematics, not only as an indispensable tool for the scientist and engineer but as a worthwhile discipline in itself, the indications are that the trends in evidence since the middle 1950's will continue. The increasing opportunities for employment for mathematically trained persons in industry, government, and teaching at all levels will attract even larger numbers of students. Expanding graduate programs will produce larger numbers of degree holders at the master's and doctorate levels. It is quite apparent that there will be a real problem to staff adequately in mathematics and statistics the institutions which are expanding not only in number but in size as well.

TABLE 3

TOTAL NUMBER OF DOCTORATES IN MATHEMATICS AND STATISTICS AWARDED BY INDIVIDUAL INSTITUTION: 1949–50 THROUGH 1958–59

Rank Order	Institution	10-yr. total	Rank Order	Institution	10-yr. total
1	University of California (all campuses)	170	36	Syracuse University	18
2	New York University	120	36	Virginia Polytechnic Inst.	18
3	Princeton University	114	39	University of Notre Dame	16
4	University of Chicago	101	39	University of Kentucky	16
4	University of Michigan	101	39	University of Tennessee	16
6	University of Illinois	92	39	George Washington University	16
7	Columbia University	82	43	Rice University	15
8	Harvard University	71	43	Catholic University of America	15
8	Massachusetts Inst. of Tech.	71	45	Illinois Institute of Technology	14
10	Brown University	70	45	Northwestern University	14
11	University of North Carolina, Chapel Hill	68	47	University of Georgia	13
12	University of Wisconsin	57	48	University of Missouri	12
13	Iowa State University	55	48	University of So. California	12
14	Purdue University	54	48	University of Oklahoma	12
15	Yale University	50	48	Oregon State College	12
16	University of Minnesota (all campuses)	48	52	George Peabody College for Teachers	11
17	Stanford University	47	52	University of Virginia	11
18	University of Pittsburgh	43	54	University of Colorado	10
18	University of Texas	43	55	Boston University	9
20	Carnegie Inst. of Tech.	40	55	University of Cincinnati	9
21	University of Florida	34	55	Washington University	9
21	University of Pennsylvania	34	58	University of Buffalo	8
23	University of Washington	33	59	Louisiana State University	7
24	Calif. Inst. of Tech.	32	59	Johns Hopkins University	7
25	University of North Carolina, State College	31	59	University of Rochester	7
26	Indiana University	29	59	Case Institute of Technology	7
26	Ohio State University	29	63	Auburn University	6
28	Cornell University	26	63	Oklahoma State University	6
29	Duke University	24	63	Pennsylvania State University	6
30	Tulane University	23	63	Vanderbilt University	6
30	University of Oregon	23	67	Rutgers, The State University	5
32	University of Maryland	22	67	Lehigh University	5
33	State University of Iowa	21	69	University of Utah	4
34	University of Kansas	19	69	American University	4
34	Michigan State University	19	71	Wayne State University	3
36	St. Louis University	18	71	University of Nebraska	3
			71	Radcliffe College	3
			74	Rensselaer Polytechnic Institute	1
			74	Bryn Mawr College	1

CONTEMPORARY MATHEMATICS*

The following report presents the uses of the two courses presented by Continental Classroom in the Contemporary Mathematics series—*Modern Algebra* in the first semester and *Probability and Statistics* in the second semester—during the academic year 1960–61.

Both courses were utilized by a variety of groups, including: (1) colleges and universities, (2) high schools, (3) local boards of education, (4) industrial organizations, (5) vocational organizations, and (6) the general public.

Colleges and universities. Approximately 300 colleges and universities across the country offered *Modern Algebra* (and/or *Teaching of Modern Algebra*) for credit during the first semester. In most cases, three semester hours of undergraduate credit was awarded for Dr. John L. Kelley's Monday, Wednesday, Friday lessons; similarly, graduate credit was granted for students following these lessons in addition to Dr. Julius H. Hlavaty's Tuesday, Thursday lessons—*Teaching of Modern Algebra*. The credit students numbered close to 5,000.

In the second semester, the number of participating institutions increased to 325. The same amount of credit was given, *i.e.*, three semester hours to undergraduate students following Dr. Frederick Mosteller's lectures in *Probability and Statistics* on Monday, Wednesday, and Friday, and graduate credit to those following these lessons in addition to Professor Paul Clifford's Tuesday and Thursday sessions—*Teaching of Probability and Statistics*. The number of credit students for these courses seems to be more than 5,000.

High schools. More than 100 high schools utilized *Modern Algebra*, for either credit or audit students (or, in some cases, both.) In the second semester, 66 high schools or school districts used *Probability and Statistics*. The number of students per school varies from several all the way to 250; approximately 1200 high school students are participating in the program.

Local boards of education. Many city boards of education have been offering in-service credit to their teachers for the mathematics courses. Among these are: New York City; Reading, Pennsylvania; Kansas City, Missouri; Newton, Massachusetts; Farmingdale, New York; Norwalk, Connecticut; Rochester, Minnesota; Cleveland Heights, Ohio; and, Ithaca, New York.

Industrial organizations. Industrial organizations have evinced particular interest in the *Probability and Statistics* series since it is so pertinent to their quality control and computer employees. Some of these organizations have set up formal classes in their plants with one of the employees (often a training officer) acting as teacher; others have merely encouraged their employees to watch the TV lessons. Among the formalized groups are: Western Electric Company, Winston-Salem, North Carolina and Allentown, Pennsylvania; Otarion Listener Corporation, Ossining, New York; New York Telephone Company, Brooklyn, New York; Market Research Corporation of America, New York City; *et al.*

Vocational organizations. Among the vocational organizations actively interested in the second semester offerings are the American Society for Quality Control and the American Orthopsychiatric Association. The former has done much promotion in its Journal, both in news articles and in printing of the TV Lesson Schedule. The latter, through its Committee on Research, invited LRI to participate in a joint mailing to its membership of 1,800 persons. The mailing was organized and sent out early in January.

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* See also p. 688 of this issue.

Another joint mailing—LRI and Addison-Wesley Publishing Company—was sent out early in January to the membership lists of the American Statistical Association and the National Council of Teachers of Mathematics. College teachers of mathematics were also included in this mailing.

General public. Based on letters and requests for TV Lesson Schedules that we have received, we may conclude that there is a large, noncredit audience among the general public in all parts of the country. This group includes: mathematicians, people engaged in scientific and other vocations related to mathematics, teachers and laymen who studied mathematics prior to the developments contained in the *Modern Algebra* and *Probability and Statistics* courses, "students" not seeking a degree, housewives, businessmen, etc.

These are the uses of the Contemporary Mathematics series to date. There will be future use and lasting value due to the availability of kinescopes made from the video tapes of the lectures. Also, it is likely that the courses will be re-run over network and educational facilities.

(Report from Learning Resources Institute. The courses are sponsored by the Conference Board of the Mathematical Sciences.)

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1476. *Proposed by M. T. Salhab, Illinois Institute of Technology*

In triangle ABC, $AB=AC$, D is the midpoint of BC, E is the foot of the perpendicular from D on AC, and F is the midpoint of DE. Prove that AF is perpendicular to BE.

E 1477. *Proposed by Sidney Kravitz, Dover, New Jersey*

It is known that $f(n)=n^2-n+41$ yields prime numbers for $n=1, \cdots, 40$. Prove that (a) $f(n)$ is never divisible by a number <41, (b) $f(n)$ is never a perfect square except for $n=41$, (c) for each n there exists an m such that $f(m)=f(n)f(n+1)$, (d) $f(1722)$ is the smallest f with four, not necessarily distinct, prime factors.

E 1478. *Proposed by Jonathan Sondow, University of Wisconsin*

Prove that the digital root of every even perfect number except 6 is 1.

E 1479. *Proposed by Morton Abramson, McGill University*

Find the number of ways of choosing k elements from n elements $x_1, \cdots, x_n$ so that no three consecutive elements appear in each choice.

E 1480. *Proposed by Leo Flatto and A. G. Konheim, IBM, Yorktown Heights, New York*

Let the line segment $[0, 1]$ be divided into $n+1$ segments by n points $P_1, \cdots, P_n$. Assume the $\{P_i\}$ are independent random variables each uniformly distributed on $[0, 1]$. What is the probability p that the $n+1$ segments can be joined to form an $(n+1)$-sided polygon?

SOLUTIONS

E 1434, Correction and Addition

E 1434 [1960, 802; 1961, 381]. *Proposed by Anatole Beck, University of Wisconsin*

Show that every open set in the plane can be represented as a disjoint union of closed straight line segments.

Editorial Note. Solutions I and II [1961, 381] are faulty. Solution I states that an open interval can be represented as the union of its closed "middle thirds," whereas there is actually a set of uncovered points of the open interval having the power of the continuum (namely the Cantor set minus a countable set). Also, the statement in Solution I that an open linear segment is the union of disjoint closed nondegenerate segments can be shown to be false. Solution II is incorrect since the unfilled part of R is *not* again a half-open rectangle, but a rectangle open on *three* sides. A correct solution along the lines of Solution II can easily be formulated, and such a solution appears below. Robert Spira has made the interesting conjecture that every open set in the plane can be represented as a disjoint union of closed straight line segments, *no two having the same direction.*

IV. *Solution by R. G. Kayel, Michigan State University.* The following construction shows how to cover the half-open square $S = \{(x, y): 0 \leqq x < 1, 0 \leqq y < 1\}$ with a union of disjoint line segments. (1) Construct the step function $S(x) = \frac{1}{2}$ if $x \in [0, \frac{1}{2}]$, $\frac{3}{4}$ if $x \in (\frac{1}{2}, \frac{3}{4}]$, $\frac{7}{8}$ if $x \in (\frac{3}{4}, \frac{7}{8}]$, $\cdots$. (2) Let the vertical segment (*VS*) $[0, \frac{1}{2}]$ move on the horizontal segment (*HS*) $[0, \frac{1}{2}]$; *VS* $[0, \frac{3}{4}]$ on *HS* $(\frac{1}{2}, \frac{3}{4}]$; *VS* $[\frac{3}{4}, \frac{7}{8}]$ on *HS* $(\frac{3}{4}, \frac{7}{8}]$; $\cdots$. (3) Let *HS* $[0, \frac{1}{2}]$ move on *VS* $(\frac{1}{2}, \frac{3}{4}]$; *HS* $[0, \frac{3}{4}]$ on *VS* $(\frac{3}{4}, \frac{7}{8}]$; $\cdots$.

The Unique and Mystic Order of Blushing Beauties

E 1446 [1961, 62]. *Proposed by David Bickerstaff, University of Mississippi*

"How about telling me confidentially the secret order of the five beauties to be featured in this year's Annual?" I proposed to the editor. She, of course, refused, but agreed to pass judgment on my guess. "Is it *A-B-C-D-E*?" I asked.

"You are most skillful at being wrong," she chided. "You not only got each person out of her true position but, furthermore, not one in your ranking followed correctly her immediate predecessor."

"Well, then, is it *D-A-E-C-B*?" I asked.

"Now you are improving," she encouraged cautiously. "You have two in proper position and you have two following correctly their immediate predecessors."

After a little figuring I then told her the correct order, and she swore me to secrecy. What is the correct order?

Solution by J. F. Leetch, Ohio State University. In the ordering *D-A-E-C-B*, the correct pair must be adjacent, for otherwise we would have a proper follower after a correct position, implying three correct positions. This means that *D-A*, *A-E*, *E-C*, or *C-B* are properly located. The pairs *A-E* and *E-C* must be eliminated since a second proper follower could not occur. The pair *D-A* can be eliminated by checking two cases. This leaves only *E-D-A-C-B* meeting all requirements.

Also solved by Philip Anderson, R. H. Anglin, Merrill Barnebey, R. E. Beals, Jeanette Bickley, A. M. Broshi, Brother Joseph Heisler, Brother T. C. Wesselkamper, Mike Brown, W. E. Buker, F. P. Callahan, Jr., Robert Carlos, Virginia Christian, Curt Gilchrist, D. I. A. Cohen, James Cooley, R. W. Cottle, Monte Dernham, R. D. Dickson, Jane Evans, D. P. Giesy, Michael Goldberg, L. D. Goldstone, S. H. Greene, W. G. Griggs, F. C. Hall, J. E. Homer, Jr., J. T. Humphrey, A. R. Hyde, Lawrence Isenecker, V. F. Ivanoff, J. L. Johnson, P. B. Johnson, Sidney Kravitz, C. W. Kreke, Betty Levine, Adolph Lu, Glen Luchau, Frank McGee, Robert Maas, D. C. B. Marsh, J. M. Mettler, Otto Mond, D. A. Moran, J. B. Muskat, Herbert Nadler, R. W. Neufeld, J. C. Nichols, C. S. Ogilvy, J. M. Pasachoff, Jon Petersen, J. L. Pietenpol, C. F. Pinzka, Stephen Porcari, E. H. Primoff, M. B. Richins, Jonathan Robinson, O. J. Roman, David Sachs, Sister Mary Denis, E. L. Spitznagel, Jr., W. B. Stovall, Jr., R. S. Strichartz, D. V. Susco, Fred Suvorov, Guy Torchinelli, J. D. Vineyard, R. M. Warten, W. C. Waterhouse, M. J. Wiedel, R. J. Wisner, and Bell Yung. Late solution by Eric Sturley.

Construction of a Triangle

E 1447 [1961, 62]. *Proposed by Walter Bluger, Dominion Bureau of Statistics, Ottawa, Canada*

Construct a triangle given R, r, h_a.

Solution by L. D. Goldstone, Watervliet, New York. The datum (h_a, r, r_a) yields r_a, then the datum $(R, r_a - r, a)$ yields a, and then the triangle is easily constructed.

For the relations yielding the above data see Altshiller-Court, *College Geometry* (2nd ed.), Sec. 139 and Sec. 144 (a). These sections, along with Sec. 140, Sec. 152, and some elementary geometry, show that the solution is essentially unique and that for compatibility we must have

$$(2R - h_a)(h_a - 2r) \geq r^2, \qquad 2R > h_a > 2r, \qquad R \geq 2r.$$

Also solved by Carole Colebob, Michael Goldberg, D. C. B. Marsh, Beckham Martin, Peter Ploch, O. J. Roman, Guy Torchinelli, and the proposer.

Goldberg showed that the distance from vertex A to the incenter is the mean proportional between $2R$ and $h_a - 2r$. The construction of the triangle now easily follows.

Marsh found constructible expressions for the sides a, b, c in terms of R, r, h_a.

Section of an Oblate Spheroid

E 1448 [1961, 62]. *Proposed by C. B. Grosch, General Mills, Inc., Minneapolis, Minnesota*

Show that any plane section of an oblate spheroid, not perpendicular to the axis of the spheroid, is an ellipse with major axis parallel to the equatorial plane of the spheroid and with minor axis (or minor axis extended) intersecting the axis of the spheroid.

I. *Solution by Amos Nannini, University of Manitoba.* With no loss of generality on account of the symmetry involved, we may assume that, the xy-plane being the equatorial plane and the z-axis the axis of revolution, the cutting plane is parallel to the y-axis. The intersection, necessarily an ellipse, again by reasons of symmetry, has one axis parallel to the y-axis and the other axis coplanar with the z-axis. To prove that the former must be the major axis, let us assume it is instead the minor. Then, by making the inclination of the cutting plane closer and closer to 90°, by reasons of continuity the intersection ought to be a circle for a definite value of the angle of inclination, because an exactly vertical plane would cut out a "squat" ellipse. But this is impossible, because an oblate spheroid has no umbilics except its poles.

The problem (and proof) can be generalized to any surface of revolution under some broad conditions of continuity and differentiability. The following might be a suitable generalization: "Any plane section of a surface of revolution has an axis of symmetry coplanar with the axis of revolution and therefore either parallel to or intersecting it, naturally or extended. Needless to say, if the curve of intersection has an axis of symmetry normal to the previous one, it must be orthogonal to the axis of revolution."

The statement of the problem as presented should be completed by mentioning the possible parallelism of the minor axis to the axis of revolution.

II. *Solution by Michael Goldberg, Washington, D. C.* A plane section of a sphere is a circle. If the sphere is deformed parallel to a diameter by a reduction of scale, the sphere becomes an oblate spheroid. The plane section remains plane and the circular section becomes an ellipse. The diameter of tne circle along the line joining the center of the circle to the axis of contraction is reduced, while the diameter of the circle normal to this line remains unchanged. Hence, by symmetry, these are the minor and major axes of the ellipse.

Conversely, any oblate spheroid can be converted into a sphere by an increase in scale parallel to the axis. A plane section of the spheroid is transformed into a plane section of the sphere.

Also solved by A. P. Boblétt, Jane Evans, D. C. B. Marsh, and the proposer. All these solutions were analytical.

A Pseudo Mean

E 1449 [1961, 62]. *Proposed by C. S. Patlak, Department of Health, Education, and Welfare, Bethesda, Maryland*

Assume that (1) A_i, B_i, C_i, D_i $(i=1, \cdots, n)$ are all positive, (2) $\sum A_i \geqq \sum C_i$, (3) $A_i - C_i = B_i - D_i$ $(i = 1, \cdots, n)$. Set $P_i = A_iB_i/C_iD_i$ and $P=\max(P_1, \cdots, P_n)$. Prove that $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i) \leqq P$.

Solution by D. C. B. Marsh, Colorado School of Mines. The proposition is obviously true for $n=1$.

Assume that $n=2$ and set $A=A_1+A_2$, $B=B_1+B_2$, $C=C_1+C_2$, $D=D_1+D_2$. Consider $P-AB/CD$, which may be written in the form

$$(1/4CD)\{4(P-1)CD + (C+D)^2 - (A+B)^2\}.$$

Since $P \geqq A_iB_i/C_iD_i$ $(i=1, 2)$, we have

$$\{4(P-1)C_iD_i + (C_i+D_i)^2\}^{1/2} \geqq A_i + B_i > 0.$$

Denoting the radical by R_i we have

$$\begin{aligned} P-AB/CD &\geqq (1/4CD)\{4(P-1)CD+(C+D)^2-(R_1+R_2)^2\} \\ &\geqq (1/4CD)\{4(P-1)(C_1D_2+C_2D_1)+2(C_1+D_1)(C_2+D_2)-2R_1R_2\}, \end{aligned} \tag{1}$$

when we expand $(R_1+R_2)^2$ and simplify algebraically. Since $A \geqq C$ it follows that, for some i, $A_i \geqq C_i$. For this i we also have $B_i \geqq D_i$, whence $P_i \geqq 1$, or $P \geqq 1$. Thus

$$4(P-1)(C_1D_2 + C_2D_1) + 2(C_1 + D_1)(C_2 + D_2) + 2R_1R_2 > 0. \tag{2}$$

Multiplying both sides of inequality (1) by the positive expression (2), we find that the right-hand side reduces to $(1/4CD)\{16P(P-1)(C_1D_2-C_2D_1)^2\}$, which is nonnegative. Thus the left-hand side must also be nonnegative, whence $P-AB/CD \geqq 0$, proving the proposition for $n=2$.

Now assume as an induction hypothesis that the proposition is true for $n=k>2$, and set

$$A^* = \sum_{i=1}^{k} A_i, \qquad B^* = \sum_{i=1}^{k} B_i, \qquad C^* = \sum_{i=1}^{k} C_i, \qquad D^* = \sum_{i=1}^{k} D_i.$$

Then, applying the proposition for $n=2$, we have

$$\begin{aligned} (A^* + A_{k+1})(B^* + B_{k+1})/(C^* + C_{k+1})(D^* + D_{k+1}) &\leqq \max[A^*B^*/C^*D^*, A_{k+1}B_{k+1}/C_{k+1}D_{k+1}] \\ &\leqq \max[\max(P_1, \cdots, P_k), P_{k+1}] = \max[P_1, \cdots, P_{k+1}], \end{aligned}$$

and the proposition holds for arbitrary positive integral n by mathematical induction.

Also solved by the proposer, who gave examples to show that all three conditions are necessary for the theorem to be true in general. He also gave an example illustrating that the quotient $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i)$ is not a true mean of the expressions A_iB_i/C_iD_i; in order that also $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i) \geqq \min(P_1, \cdots, P_n)$, condition (2) would have to be that $\sum A_i = \sum C_i$.

Application of Abel's Partial-Summation Theorem

E 1450 [1961, 62]. *Proposed by Lawrence Shepp, Princeton University*

If $a_n, b_n > 0$, $a_n \downarrow 0$, then $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j = \sum_{n=1}^{\infty} a_n b_n$.

I. *Solution by W. C. Waterhouse, Harvard University.* We note, by Abel's partial-summation theorem,

$$S_k \equiv \sum_{n=1}^{k} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j = \sum_{n=1}^{k} a_n b_n - a_{k+1} \sum_{n=1}^{k} b_n = \sum_{n=1}^{k} (a_n - a_{k+1}) b_n.$$

Thus $\sum_{n=1}^{k} a_n b_n > S_k$, so if $S_k \to \infty$ then $\sum_{n=1}^{k} a_n b_n \to \infty$.

Suppose $\lim S_k = A$. Then $\sum_{n=1}^{\infty} a_n b_n \geqq A$. But for any k and any $\epsilon > 0$ we can pick an $m > k$ such that $a_m / a_k < \epsilon$. Then

$$A \geqq S_{m-1} \geqq \sum_{n=1}^{k} (a_n - a_m) b_n > (1 - \epsilon) \sum_{n=1}^{k} a_n b_n,$$

whence $A \geqq \sum_{n=1}^{\infty} a_n b_n$. It follows that $A = \sum_{n=1}^{\infty} a_n b_n$.

II. *Solution by Michael Goldberg, Washington, D. C.* Consider the step function of vertices $(0, a_1)$, (b_1, a_1), (b_1, a_2), $(b_1 + b_2, a_2)$, $\cdots$, $(\sum_{j=1}^{k} b_j, a_j)$, $(\sum_{j=1}^{k} b_j, a_{j+1})$, $\cdots$. Then, if the area A under this function is divided into vertical strips, it is given by $A = \sum_{n=1}^{\infty} a_n b_n$, while if it is divided into horizontal strips, it is given by $A = \sum_{n=1}^{\infty} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j$. Hence the desired result follows.

Also solved by J. L. Brown, Jr., P. R. Chernoff, Gus DiAntonio, S. H. Greene, Betty Levine, Jiang Luh, D. C. B. Marsh, J. M. Pasachoff, J. T. Rosenbaum, O. E. Stanaitis, Fred Suvorov, and the proposer. Late solution by Eric Sturley.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4977. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let S be a system with a finite number of elements with two operations, addition and multiplication, satisfying the following axioms:

1) Under addition the elements of S form an abelian group.
2) For a, b, c in S, $a \neq b$, the equation $xa = xb + c$ has a unique solution.
Prove that also the equation $ay = by + c$ has a unique solution.

4978. *Proposed by R. Nathan, University of Idaho*

Let $n > 0$ and $k \geqq 2$ be integers. Show

$$\sum_{\mu=0}^{n} \frac{(-1)^{\mu}}{\mu![(\mu+1)k-1](n-\mu)!} = \frac{k^n}{(k-1)(2k-1)\cdots([n+1]k-1)}.$$

4979. *Proposed by H. S. Shapiro, New York University*

If z_1, z_2, z_3 are distinct numbers of modulus 1, and

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1^m & z_2^m & z_3^m \\ z_1^n & z_2^n & z_3^n \end{vmatrix} = 0,$$

then either two rows or two columns of the determinant are identical.

4980. *Proposed by G. H. Meisters, RIAS, Baltimore, Md.*

If P is a topological property, we call a topological space X *locally-P* if and only if every neighborhood N of every point x contains a neighborhood N^* of x which has the property P in its relative topology. Otherwise our terminology is that of J. L. Kelley: *General Topology*. Prove (or disprove) the following statement: If X is a compact, locally-Hausdorff topological space, then X is locally-compact.

4981. *Proposed by Lawrence Shepp, Princeton University*

Show

$$\int_0^1 \left(\sum_{n=1}^{\infty} \{2^{-n}(1 - \cos 2^{2n}z)\} \right)^{-1} dz < \infty.$$

4982. *Proposed by G. Di Antonio, Duquesne University*

On any of the five regular solids, let two points be given, not both on the same face. Determine the geodesic between the two points.

SOLUTIONS

A Corollary of a Theorem of Schwartz

4906 [1960, 479]. *Proposed by P. L. Butzer, Technical University, Aachen, Germany*

If the function $f(x)$ is continuous on an interval (a, b) and, as $h \to 0$, $h^{-2}\int_0^h [f(x+u)+f(x-u)-2f(x)]du \to 0$ for all x in (a, b), then $f(x)$ is a linear function.

II. *Solution by Joshua Barlaz, Rutgers—The State University.*

Let $[\alpha, \beta]$ be a closed interval in (a, b). With x in $[\alpha, \beta]$, k real, let

$$\psi(x) = f(x) + \frac{\beta - x}{\alpha - \beta} f(\alpha) + \frac{x - \alpha}{\alpha - \beta} f(\beta) + k^2(x - \alpha)(x - \beta).$$

Then $\psi(x)$ is continuous on $[\alpha, \beta]$ and, as $h \to 0$,

$$(1/h^3) \int_0^h [\psi(x + u) + \psi(x - u) - 2\psi(x)]du \to 2k^2/3 > 0.$$

Suppose now that the maximum of $\psi(x)$ occurred at $x = \xi$ and $\alpha < \xi < \beta$. Then, for sufficiently small u,

$$\psi(\xi + u) + \psi(\xi - u) - 2\psi(\xi) \leqq 0$$

and the limit above could not be positive. Therefore the maximum of $\psi(x)$ is either at $x = \alpha$ or at $x = \beta$. But $\psi(\alpha) = \psi(\beta) = 0$. Consequently $\psi(x) \leqq 0$. Letting $k \to 0$, it follows that

$$f(x) + \frac{\beta - x}{\alpha - \beta} f(\alpha) + \frac{x - \alpha}{\alpha - \beta} f(\beta) \leqq 0, \qquad \alpha \leqq x \leqq \beta.$$

A parallel argument also proves

$$f(x) + \frac{\beta - x}{\alpha - \beta} f(\alpha) + \frac{x - \alpha}{\alpha - \beta} f(\beta) \geqq 0$$

and therefore $f(x)$ is linear on $[\alpha, \beta]$. But α, β are arbitrary in (a, b). Therefore $f(x)$ is linear on (a, b).

This proof parallels that given for Schwartz's theorem in Chaundy, *The Differential Calculus.*

Note by H. W. Brinkmann, Swarthmore College. The solution previously given [1961, 385] is incorrect since it uses, not l'Hospital's rule, but a converse of it which is notoriously not true. The presumed theorem (*If $f(x)$ and $g(x)$ are of class C' and $f(x)/g(x) \to L$ as $x \to a$, then $f'(x)/g'(x) \to L$*) is easily disproved by the counter example: $f(x) = x^4 \sin(1/x)$, $g(x) = x^3$, $(a = 0)$.

Arguments of Two Sets of Complex Numbers

4918 [1960, 699]. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let α_k, β_k be complex numbers, $k = 1, \cdots, n$, with $|\alpha_k| = |\beta_k| = 1$. Let $0 \leqq \arg \alpha_1 < \cdots < \arg \alpha_n < 2\pi$, $0 \leqq \arg \beta_1 < \cdots < \arg \beta_n < 2\pi$, where $0 \leqq \arg \gamma < 2\pi$ for any nonzero complex number γ. Also let $\sum_1^n \alpha_k = \sum_1^n \beta_k = 0$. Prove that $\sum_1^n \alpha_k \bar{\beta}_k \neq 0$.

Editorial Note. A solution by P. J. van Albada, G. Laman, and J. H. van Lint appears in the *Michigan Journal of Mathematics*, vol. 7, 1961.

A Definite Integral Property

4919 [1960, 699]. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let $\phi(\theta)$ be real valued for $0 \leqq \theta \leqq 2\pi$, let $\phi'(\theta)$ be continuous and positive, let $\phi(0) = 0$, $\phi(2\pi) = 2\pi$ and let $\int_0^{2\pi} e^{i\phi(\theta)} d\theta = 0$. Prove that

$$\left| \int_0^{2\pi} e^{i(\phi(\theta)-\theta)} d\theta \right| > 4.$$

Editorial Note. No proof of the problem as stated has been received. However, the proposers have solved it with an additional hypothesis on $\phi(\theta)$. This will appear in an early issue of the *Michigan Journal of Mathematics.*

Number of Games Won in a Major League Baseball Season

4921 [1960, 700]. *Proposed by David Gale, Brown University*

Prove that the numbers $w_1, \cdots, w_8$ can be the numbers of games won by the eight teams at the end of a major league baseball season if and only if

$$\sum_{i=1}^{8} w_i = 616, \qquad \sum_{i=1}^{k} w_i \leqq 11k(15-k), \qquad k = 1, \cdots, 7.$$

(During a major league season every two teams play each other 22 times.)

Solution by D. C. B. Marsh, Colorado School of Mines. Assuming no ties and no incompleted games, the necessity of the given conditions is easily shown. Putting w_{ij} for the number of times i won from j; $w_{ii} = 0$ and $w_{ij} + w_{ji} = 22$ for all i, j with $i \neq j$, we have

$$\sum_{i=1}^{k} w_i = \sum_{j=1}^{8} \sum_{i=1}^{k} w_{ij} = \sum_{j=1}^{k} \sum_{i=1}^{k} w_{ij} + \sum_{j=k+1}^{8} \sum_{i=1}^{k} w_{ij}.$$

Now $\sum_{i=1}^{k} \sum_{j=1}^{k} w_{ij} = 22C(k, 2) = 11k(k-1)$ and $0 \leqq \sum_{j=k+1}^{8} \sum_{i=1}^{k} w_{ij} \leqq 22(8-k)k$, whence

$$(3) \qquad 11k(k-1) \leqq \sum_{i=1}^{k} w_i \leqq 11k(k-1) + 22k(8-k) = 11k(15-k),$$

for $k = 1, \cdots, 8$, which implies the given conditions.

The conditions as stated are not sufficient, for they are satisfied by $w_1 = 154$, $w_2 = \cdots = w_7 = 0$, $w_8 = 462$, which nevertheless contradict $w_{23} + w_{32} = 22$, etc.

Also solved by John E. Freund, David Greenstein, Sidney Kravitz; and also (necessity only) J. R. Blum, J. B. Bohac, G. S. Cunningham, G. DiAntonio, Jane Evans, D. L. Muench, Martha M. Pennell, D. C. Stevens, and Eric Sturley.

Editorial Note. If we relabel the w's so that $w_1 \geqq w_2 \geqq \cdots \geqq w_8$, the left member of (3) can be replaced by $77k$. It is then not difficult to show that the modified inequalities provide also sufficient conditions. One way of establishing this result is to consider n teams each playing 22 games with each of the others. Then (3) becomes $11k(n-1) \leqq \sum_{i=1}^{k} w_i \leqq 11k(2n-1-k)$, $k = 1, \cdots, n$. The sufficiency is obvious for $n = 2$, and we then show that sufficiency for n follows from sufficiency for $n-1$.

Abelian Group without Maximal Subgroups

4922 [1960, 700]. *Proposed by Peter Crawley, California Institute of Technology*

In the solution of a recent problem (4761, [1959, 67]) it was shown that the additive group of rationals has no maximal subgroups. Prove the generalization: an abelian group G has no maximal subgroups if and only if G is divisible (*i.e.*, $NG=G$ for all integers n).

Solution by Paul Hill, Institute for Advanced Study. If H is a proper subgroup of a divisible abelian group G, then G/H is divisible and therefore cannot be cyclic. Thus H is not maximal in G. To prove the converse, use is made of the following obvious propositions: (1) if H is a subgroup of an abelian group G having no maximal subgroups, then G/H has no maximal subgroups, (2) a group G having a cyclic summand has a maximal subgroup unless G is prime order cyclic. Now suppose that the abelian group G is not divisible. The quotient group G/nG is nontrivial for some positive integer n. If G/nG is prime cyclic, then nG is maximal in G; otherwise, G/nG contains a maximal subgroup since from the well-known first theorem of Prufer it follows that G/nG is a direct sum of cyclic groups. Thus G contains a maximal subgroup.

Also solved by T. N. Delmer, C. Franke, E. R. Gentile, Nathaniel Grossman, J. M. Irwin, S. Lajos, R. A. McHaffey, Fred Suvorov, and the proposer.

Editorial Note. The stated proposition is equivalent to: *An Abelian group G is divisible if and only if $\phi(G)=G$, where $\phi(G)$ is the Frattini subgroup of G*, part of an exercise in L. Fuchs, *Abelian Groups*, 1958, p. 67.

Gentile proves the following generalization: *Let R be a Dedekind ring. Then an R-module is divisible if and only if it has no maximal submodules.*

The Divisor Set of a Set of Real Numbers

4923 [1960, 808]. *Proposed by D. J. Newman, Yeshiva University*

For any set of real numbers S we define its divisor set $\hat{S}$ to be the set of all numbers in (0, 1) for which some integral multiple lies in S. Are there sets of arbitrarily large measure for which $\hat{S}$ has arbitrarily small measure?

Solution by J. B. Kelly, Michigan State University. The following example provides an affirmative answer. Let

$$S = \sum_{k=1}^{[n/\epsilon]} \left(\frac{n}{k}, \frac{n+\epsilon}{k} \right).$$

Here n is a positive integer and ϵ is a positive number less than unity whose dependence upon n will be given later. The intervals comprising S are disjoint since $n/k<(n+\epsilon)/(k+1)$ implies $k>n/\epsilon$. Hence, denoting the measure of S by $m(S)$, we have

$$m(S) = \sum_{k=1}^{[n/\epsilon]} \epsilon/k > \epsilon \log (n/\epsilon).$$

The divisor set $\hat{S}$ is the same as the divisor set for the open interval $(n/k, (n+\epsilon)/k)$. The divisor set for the latter consists of intervals $(n/k, (n+\epsilon)/k)$ where k runs from $n+1$ to $[n/\epsilon]$, plus the interval from 0 to ϵ. Hence, using the fact that $\sum_{k=m}^{rm} 1/k \sim \log r$, we have

$$m(\hat{S}) = \epsilon + \sum_{k=m+1}^{[n/\epsilon]} \epsilon/k < \epsilon + 2\epsilon \log(1/\epsilon)$$

for n sufficiently large. Now, setting $\epsilon = (\log n)^{-1/2}$ and letting $n \to \infty$, we see that $m(S) \to \infty$ and $m(\hat{S}) \to 0$.

Also solved by Robert Breusch, Fred Suvorov, and the proposer.

A Pseudo-integral

4924 [1960, 808]. *Proposed by J. H. Blau, Antioch College*

Let P denote a partition of $[0, 1]$ into n disjoint measurable sets E_i. Denote $\sup m(E_i)$ by $|P|$, and let $x_i \in E_i$. For which functions f on $[0, 1]$ does the pseudo-integral exist:

$$\lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i) m(E_i)?$$

Solution by the proposer. The integral exists and equals J if and only if those values of x (if any) for which $f(x) \neq J$ form a sequence $\{a_i\}$, and $f(a_i) \to J$. For the proof, let $J = 0$. The extension to general J is clear.

Let the integral exist. Suppose $f(x) \geqq \epsilon > 0$ for infinitely many distinct values b_i of x. Then, for any $\delta > 0$, there is a partition P with $|P| < \delta$, and with $b_i \in E_i$. Thus $\sum_{i=1}^{n} f(b_i) m(E_i) \geqq \epsilon$, contradicting the integral's existence. Hence $f(x) < \epsilon$ with a finite number of exceptions. The same is true for $f(x) > -\epsilon$, and hence for $|f(x)| < \epsilon$. It follows that the set where $f(x) \neq 0$ is denumerable. Enumerating it as $\{a_i\}$, we have $f(a_i) \to 0$.

Conversely, let the condition be satisfied. Then $M = \sup |f|$ is finite. If $\epsilon > 0$, then $2|f(x)| \geqq \epsilon$ for only a finite number, say r, of values of x. Let P be such that $2rM|P| < \epsilon$, let $x_i \in E_i$, and renumber the E_i so that $2|f(x_i)| < \epsilon$ for $i > r$. Now

$$\left| \sum_{i=1}^{n} f(x_i) m(E_i) \right| \leqq \sum_{i=1}^{n} |f(x_i)| \, m(E_i).$$

The sum of the first r terms on the right is at most $rM|P|$, and the remaining sum is less than $\epsilon/2$. Hence the integral exists.

Also solved by G. A. Heuer, and Fred Suvorov.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Zbornik Matematičkih Problema, III. D. S. Mitrinović (Ed.). Univerzitet u Beogradu, Beograd, 1960. xvi+334 pp. Purchasable through La Faculté d'Électrotechnique, Département Mathématique, Université de Belgrade. Zbornik I, $7.00; Zbornik II, $8.00; Zbornik III, $8.00.

With a revised edition of Volume II appearing last year and Volume I currently undergoing its third revision, this paper-bound work by the well-known European problemist presents a new set of some 870 problems to supplement the earlier pair. The main sources of material are *Jahresbericht der Deutschen Mathematiker Vereinigung*, *The American Mathematical Monthly*, *The Mathematical Gazette*, *Mathematics Magazine*, and *Mathesis*, as well as original results obtained by Mitrinović and his associates.

The chapter headings give some indication of the scope of subject matter. After an introduction and a generous table of symbols and notation, we have complex numbers and functions, special functions (Legendre, Bessel, Laguerre), abstract algebra, projective geometry, miscellaneous problems, and an appendix touching on such topics as stereographic projection and generalized Jensen formulas. The work is concluded with twenty-six pages of tables (Kelvin functions, binomial coefficients, zeros of $J(x)$ and others).

The section on abstract algebra, for example, is a veritable handbook, developing semi-group and group concepts through exercises on properties and examples (matrices, permutations) up to rings and fields, and including isomorphism, ordering, *et al.*

With or without the assistance of a bilingual dictionary, one may rely upon the universal nature of the symbolism and technical terms to be able to interpret most of the content. Still there is enough expository material to warrant an English translation and one may hope that the National Science Foundation will include these volumes among those Yugoslavian works to be translated.

The printing and quality of the materials used are simple but adequate. All three volumes are recommended particularly to those interested in modern problems *per se*.

D. C. B. MARSH
Colorado School of Mines

Introduction to Analytic Geometry and Linear Algebra. By Arno Jaeger. Holt, Rinehart and Winston, New York, 1960. xii+305 pp. $6.00

This is an attempt to "present a modern . . . and a somewhat novel treatment of fundamental analytic geometry based on groups, vector spaces, and

Euclidean vector spaces together with an introduction to these algebraic structures at a level suitable for freshmen and sophomores."

The 23 chapters are divided into four major sections: I. *Foundations* introduces set theory, groups, vector spaces, and establishes the connection between algebra and geometry in terms of 1-1 onto maps between point-space $\mathfrak{R}^n$ and vector-space of translations $\mathbf{R}^n$, between $\mathbf{R}^n$ and coordinate space of n-tuples R^n, and between $\mathfrak{R}^n$ and R^n, where n is 1 or 2 or 3. II. *Linear Geometry and Algebra* concerns systems of linear equations, bases and dimension of vector spaces, linear maps, matrices. There is a nice digression on linear programming. III. *Multilinear Geometry and Algebra* introduces the inner product and deals with metric notions. Euclidean vector space is defined and put in context. Basic properties of determinants are developed. IV. *Quadratic Geometry and Algebra* deals with conics, quadrics, and canonical forms.

Demands on verbal ability and ability to handle symbols will make this book hard reading for freshmen and sophomores. There is some tendency to define concepts in greater generality than the main theme of the text requires, but in any event, the deliberately high level of abstraction requires introduction of an extensive technical vocabulary. The exposition is often quite condensed. An excessive use of italics for emphasis becomes quite tiresome, and what is more serious, reduces the effectiveness of italics in all uses.

The reviewer feels that for undergraduates having some prior college mathematics this book can be the basis for a very useful course. The many connections with geometry are welcome in an introductory algebra course. The organization is good. Theorems are well stated, definitions accurate. Abundant illustrative examples are well chosen. Lists of exercises are generally excellent. There is an adequate index supplemented by a useful index of symbols and another of examples.

ROBERT M. EXNER
Syracuse University

Complex Variable and the Laplace Transform for Engineers. By Wilbur R. LePage. McGraw-Hill, New York, 1961. xvii+475 pp. $12.50.

This is an excellent book for the purpose for which it is intended and rather than dwell on the technical qualities of the book (which are really good), the reviewer feels it is the spirit of the book which is important and should be discussed. The author is a professor of engineering and obviously feels that a properly trained engineering student is one who does his work creatively and forgets the quest for the "magic formula." This means that the student must have a real understanding of the basic area from which he draws his tools (*i.e.*, mathematics), not only from the point of view of selecting tools (theorems), when possible, to solve a problem, but also having sufficient understanding in order to create additional tools (when they are needed) with a feeling of confidence. The author must feel that the best way for the student to get to this position is to understand what definitions, theorems, and proofs are. It might

be noted at this point that he does an outstanding job of motivating his point of view. For example, in discussing the Fourier and Laplace integral, he essentially makes a good point concerning the mathematical idealization of system for analysis and the transforming of the ideal system into the construction of a practical engineering image. Hence, in the first situation the Fourier integral is effective (since energy considerations are important here) and in the second case (which is the important case for the practitioner) the Laplace integral is effective since the physical synthesis and realization of the idealized system inevitably requires the use of complex variables.

Insofar as the text itself is concerned, there are fifteen chapters. Most of the important topics found in the usual books on engineering mathematics are developed from scratch in a manner that is intellectually honest. Particular emphasis is given to complex function theory inasmuch as this plays a central and unifying role in the text. A large number of problems appear at the end of each chapter. Also, the author makes good use of physical problems to motivate the type of theorem he needs to develop and to illustrate that a hasty use of formal calculus doesn't always lead to the right result (if any at all).

A publicist for the book company writes on the front flap of the jacket that the text was written for the serious student, "probably" at the graduate level. He is really suffering from a case of commercial temerity since the contents and presentation of the book is in no way out of line with the recommendations made a few years ago by a select committee of the American Society for Engineering Education.

PASQUALE PORCELLI
Louisiana State University

Introductory Algebra. A College Approach. By M. D. Eulenberg and T. S. Sunko. Wiley, New York, 1961. xi+290 pp. $4.95.

The preface of this book states that it is intended for students in college who have not had sufficient preparation for college algebra or an equivalent course. It is supposed to be finished in one semester, since such students are more mature mentally than those in preparatory school. The introduction is well done. The book contains a six-page index, answers to odd-numbered problems and tables (1) of squares and cubes, square roots and cube roots and (2) of common logarithms of numbers.

It is a question as to whether a "set" is defined here or not. The authors say that we will regard a set as "any distinct collection of objects," but that it is an undefined word. Certainly the definition of a finite set is inadequate. It is not the *elements* that are less than a given fixed number, but the *number* of the elements.

The authors are very good in their explanation of division by zero, in many sets of problems and on functional relationships. The reviewer is glad to see the material on sets in the book. It is developed in an interesting manner.

It is distressing to find frequent use of a singular subject with a plural verb, as well as a split infinitive, the term "consecutive numbers" and a painful collection of numerical errors.

The student who uses this book will certainly know how to deal with fractions, though he may be halted temporarily when told (on p. 128) that $18-6=24$ and (at the top of p. 134) that one root is 3/4 when it really is 4/3. The book seems to have been somewhat carelessly proofread.

MARION E. STARK
Wellesley College

Intermediate Algebra. By F. J. Mueller. Prentice-Hall, Englewood Cliffs, N. J., 1960. 374 pp. $5.95.

Written in traditional style, this book is developed by means of rules. Over seventy of these rules are included in the book, each of which is illustrated by several step-by-step examples, and followed by a large number of problems.

There is occasional mention of a term used in modern mathematics, although little effort is made to incorporate this language into the development. In the first chapter (From Arithmetic to Algebra) the principles of closure, commutativity, and associativity for addition and multiplication of a system of numbers are stated, along with the distributive principle for multiplication with respect to addition. These are illustrated with arithmetic examples but *are not used for anything.* The distributive principle is the only one of them mentioned in later chapters.

Included in the book are techniques for dealing with all the traditional problems of elementary algebra. Mr. Mueller is so clear in his statements of "how to do it" that an average student could learn how to solve the problems of this book with little extra help from a teacher. Whether he would gain any understanding of algebraic proof is questionable.

VIRGINIA CARLTON
Centenary College

Theoretical Physics in the Twentieth Century; A Memorial Volume to Wolfgang Pauli. By M. Fierz and F. F. Weisskopf (eds.). Interscience, New York, 1960. x+328 pp. $10.00.

Fourteen papers, including a foreword by Niels Bohr and a bibliography of Pauli's works by C. P. Enz, range over topics and fundamental problems close to Pauli's interests in quantum theory, relativity, and mathematical physics. Important events in research during the 1930's are also among the subjects discussed by such contributors from Europe, Russia, and the United States as W. Heisenberg, L. D. Landau, B. L. van der Waerden, G. Wentzel, C. S. Wu, and others.

TOM SMITH
University of Oklahoma

Silhouette Mathematics. By R. S. Underwood. Texas Technological College Bookstore, Lubbock, Texas, 1961. iv+148 pp. $2.00.

This text, which is paper bound and photographically reproduced from typewritten pages, largely contains the present status of the author's researches of the important and rather neglected problem of picturing the equations of three, and higher, dimensional analytic geometry on a plane. The material is elementary and merits examination. With some give and take, the text can be used in a beginning course in analytic geometry, though it would better serve a three-semester-hour tailor-made course. There are ample exercise sets, and answers are furnished in the rear of the book. There is no index, but there is a good table of contents. In the last chapter various "loose ends" and possible areas of application (such as to the solution of some Diophantine equations) are pointed out. The theory is not complete and there is room for improvements and extensions. Indeed, a dozen or so master's theses have originated with the ideas of this text.

HOWARD EVES
University of Maine

The Simplex Method of Linear Programming. By F. A. Ficken. Holt, Rinehart and Winston, New York, 1961. 64 pp. $1.50.

In this excellent little book, the author gives a clear exposition directed primarily to the mathematical theory of the simplex method of linear programming which falls within the field of algebra for the consideration of the general solution of a system of linear equations having certain constraints associated with them. The purpose appears to be an introduction to the subject in that, as is indicated by the author, there are many omissions. These, in the reviewer's opinion, are not at all detrimental. Also, the omissions are to a large extent neutralized by a rather well selected Bibliography. The reviewer also opines that one additional reference should have been mentioned, namely: *Mathematical Methods of Operations Research*, by T. L. Saaty, New York, 1959.

Ficken's book will give a reader with moderate maturity in mathematics a very intelligent idea of the basis, nature and uses of the simplex method of linear programming. The professional engineer and the business specialist as well as others who have need for linear programming will find this book very useful and informative. The topics treated have been wisely selected and their treatment is orderly and clear.

It is very gratifying to see such a lot of useful information put in a clear and understandable fashion in such a concise way. In my opinion, the author is to be congratulated and commended for a piece of work well done.

FRANK M. WEIDA
The George Washington University

Modern Fundamentals of Algebra and Trigonometry. By Henry Sharp, Jr. Prentice-Hall, Englewood Cliffs, N. J., 1961. ix+340 pp. $6.50.

This book gives an excellent treatment of basic topics in algebra and trigonometry. It emphasizes an axiomatic approach in the development of the number system and introduces set terminology before treating the concept of a function. Proofs of the binomial theorem, DeMoivre's theorem and important results in the theory of equations demonstrate the usefulness of the induction axiom. Trigonometry proceeds from the solution of right and oblique triangles to a careful treatment of trigonometric and inverse trigonometric functions. Throughout the book graphical methods are stressed—in the study of inequalities, in the analytic geometry of the straight line, in graphs of polynomials and transcendental functions, in presenting statistical data. In a chapter on theory of equations the author defines the slope of a polynomial function and thus paves the way to the study of calculus. Answers to most odd-numbered exercises are given. Appendices on inequalities, roots of polynomials and binomial series add to the worth of the book, as do various tables. The latter include trigonometric values for angles in degrees and for numbers, 4-place common logarithms and logarithms of trigonometric values. The book admirably combines a modern approach to underlying principles of mathematics with a detailed and rigorous study of algebra and trigonometry.

HELEN G. RUSSELL
Wellesley College

Strukturtheorie der Wahrscheinlichkeitsfelder und -Räume. By Demetrios A. Kappos. Springer-Verlag, Berlin, Göttingen, and Heidelberg, 1960. 136 pp. About $5.50.

The author points out that since there already exist systematic treatises on probability theory from the measure-theoretic point of view, this book does not attempt such a presentation. The book thus confines itself, as the title implies, to the systematic presentation of the mathematical theory of the structure of probability fields and, in terms of these, to that of the structure of probability spaces. The former are defined in terms of a probability function and the latter in terms of measure, but since it is shown that every probability space may be represented as a probability field, the former approach permits a more direct formulation of problems that have been treated by the latter.

The latter part of the book includes material on cartesian products and concepts of independence that play a dominant role in the characterization of the structure of probability fields. A theory of the nonseparable invariant extensions of Lebesque measure related to concepts of independence of Kakutani and Oxtoby, and some treatment of a generalization of the concept of probability space due to Renyi, appear at the end.

A. A. GRAU
Oak Ridge National Laboratory

Elementary Algebra for College Students. By Irving Drooyan and William Wooton, Wiley, New York, 1961. 272 pp. $4.95.

This book, which is traditional in its approach, assumes a knowledge of arithmetic only. It would be helpful to a student with no previous experience with algebra or to a student engaged in self-instruction in algebra. Algebraic concepts are developed intuitively but logically, and algebraic terminology is made to contribute to understanding rather than to confusion. Each new concept, including "word problems," is carefully illustrated with sample problems. Word problems early become an integral part of the text. The entire text should be covered in a first course in elementary algebra.

TRUMAN WESTER
Federal Aviation Agency Training Center, Oklahoma City

Integral Quadratic Forms, Cambridge Tracts in Mathematics and Mathematical Physics no. 51. By G. L. Watson. Cambridge University Press, New York, 1960. 143 pp. $5.00.

The author has brought together in a well-organized and well-written fashion the basic theory of quadratic forms with integral coefficients and variables. Three classes of problems are identified and studied. First there is the problem of the equivalence of quadratic forms under linear integral transformations of determinant ± 1. Proper equivalence, where the determinant is restricted to $+1$, is not introduced; this is just as well because it is strange to have x^2+2y^2 not equivalent to $2x^2+y^2$. Second, the author is concerned with decomposition; a form f in n variables is said to be decomposed into $g+h$, where g and h are forms in r and $n-r$ variables, if f is equivalent to $g+h$. The third problem is the representation of integers by forms; an integer k is represented by $f(x_1, \cdots, x_n)$ if the equation $f=k$ can be solved in integers $x_1, \cdots, x_n$. The background material needed for the study of the book is not great, nothing more than elementary number theory and an acquaintance with Dirichlet's theorem on the infinitude of primes in an arithmetic progression. However, the pace is fairly fast and sophisticated, so that typical undergraduates in universities in the U. S. would not find it easy going.

While the book for the most part constitutes a compilation and synthesis of known results, it contains some theorems by Watson not previously in the literature. The book is a felicitous addition to the small collection of such works available. The Carus monograph on the subject, *The Arithmetic Theory of Quadratic Forms* by Burton W. Jones, uses a different approach on many questions, and the emphasis is not the same. About the only other book in English on the subject is L. E. Dickson's *Studies in the Theory of Numbers*, and it is much more specialized in content than the works by Watson and Jones.

IVAN NIVEN
University of Oregon

Boolean Algebra and its Applications. By J. Eldon Whitesitt. Addison-Wesley, Reading, Mass., 1961. x+182 pp. $6.75.

One of the trying times in the life of a mathematics teacher occurs when an eager young electrical engineering student steps up at the end of a calculus lecture to ask "what do you know about this new subject called Boolean algebra?" Safe answers range from "nothing" to "a little, but it's not really in my field." Whether we like it or not, the elementary theory of Boolean algebras pops up in a remarkably wide variety of applications. It is comforting to know that there is now a book—Whitesitt's *Boolean Algebra and its Applications*—which treats the subject with clarity, care and honesty.

A look at the chapter headings in Whitesitt's book shows the range of material which it covers. In order, these are: "The algebra of sets, Boolean algebra, symbolic logic and the algebra of propositions, switching algebra, relay circuits and control problems, circuits for arithmetic computation, and introduction to probability in finite sample spaces." None of these topics are treated in detail. Nevertheless, a beginning student will find each chapter an attractive introduction to the ideas and tools of the subject with which it deals.

In the smooth, logical development of subject matter, the careful statement and explanation of definitions, the detailed presentation of examples, and the extensive offering of exercises (many of them equipped with solutions), Whitesitt's book rates a solid A. As a textbook, it should provide help and solace to students and teachers, as well as the great mass of engineers bent on self-improvement.

R. S. PIERCE
University of Washington

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Mr. R. E. Barnhill, University of Kansas, has been awarded a Woodrow Wilson National Fellowship for graduate study in the academic year 1961–62.

Professor S. R. Beyma, Hampton Institute, was named a recipient of the Christian R. and Mary F. Lindback Award for Distinguished Teaching. This award carries a $500 stipend.

Professor J. A. Brown, University of Delaware, represented the Association at the

inauguration of Dr. L. I. Mishoe as President of Delaware State College on April 16, 1961.

Professor J. M. Calloway, Kalamazoo College, represented the Association at the inauguration of Dr. J. W. Miller as President of Western Michigan University on May 20, 1961.

Professors Shiing-Shen Chern, University of California, and J. W. Tukey, Princeton University, have been elected members of the National Academy of Sciences.

Professor M. S. Hendrickson, University of New Mexico, represented the Association at the inauguration of Dr. D. C. Moyer as President of Eastern New Mexico University on May 30, 1961.

Professor Emeritus Norman Miller, Queen's University, was awarded the honorary degree of Doctor of Laws by Queen's University on May 20, 1961.

Professor Deane Montgomery, Institute for Advanced Study, received the honorary degree of Doctor of Humane Letters from Yeshiva University on June 15, 1961.

Professor A. W. Tucker, Princeton University, was awarded the honorary degree of Doctor of Science by Dartmouth College on June 11, 1960.

Agnes Scott College: Mr. R. E. R. Nelson, University of Virginia, has been appointed Instructor; Assistant Professor Sara Ripy has been promoted to Associate Professor.

Massachusetts Institute of Technology: Drs. A. E. Hurd, Norman Lebovitz and J. H. Simons have been appointed C. L. E. Moore Instructors; Drs. F. P. Bretherton, R. E. Briney, T. J. Eisler, R. L. Finney, III, R. A. Gangolli, J. P. Levine, G. J. Maltese, R. M. Moroney, M. M. Robertson, and H. C. Rumsey, Jr., have been appointed Instructors; Drs. F. M. Leslie, J. R. McCord, and Hironori Onishi have been appointed Research Associates; Assistant Professors Sigurdur Helgason, D. M. Kan, A. P. Mattuck, M. L. Minsky, and F. P. Peterson have been promoted to Associate Professors; Associate Professor Bertram Kostant, University of California and Dr. R. H. Lüst, Max Planck Institute, Munich, Gemany, have been appointed Visiting Professors; Associate Professor Felix Browder, Yale University, and Dr. Ekkehart Kröner, Stuttgart Institute of Technology, Stuttgart, Germany, have been appointed Visiting Associate Professors; Assistant Professor H. J. Weinitschke, University of California, Los Angeles, has been appointed Visiting Assistant Professor; Professor I. M. Singer is on leave on an Alfred P. Sloan Research Fellowship; Professor G. G. Whitham is on leave and will spend the year at California Institute of Technology; Associate Professor L. M. Howard is on leave at Cambridge University on a Guggenheim Fellowship; Assistant Professor K. M. Hoffman is on leave at the University of California, Los Angeles, for one year.

Rensselaer Polytechnic Institute: Associate Professors B. A. Fleishman and J. W. Hollingsworth have been promoted to Professors; Assistant Professor T. Y. Chow has been promoted to Associate Professor.

University of Arizona: Dr. J. A. Dyer, University of Texas, has been appointed Assistant Professor; Professors L. M. Milne-Thomson, Mathematics Research Center, University of Wisconsin, and L. J. Mordell, Cambridge University, Cambridge, England, have been appointed Visiting Professors; Dr. M. S. Cheema, University of California, and Dr. N. C. Giri, Stanford University, have been appointed Visiting Assistant Professors.

University of Rochester: Drs. A. H. Stone and Dorothy M. Stone, University of Manchester, Manchester, England, and Professor J. H. B. Kemperman, Purdue University, have been appointed Professors; Drs. W. W. Comfort, Harvard University, K. A. Ross, University of Washington, and C. E. Watts, Institute for Advanced Study, have been appointed Assistant Professors; Assistant Professor Yuzo Utumi, McGill University, has been appointed Visiting Assistant Professor.

Mr. E. Z. Andalafte, University of Missouri, has been appointed Instructor at Southwest Missouri State College.

Dr. J. M. Anderson, University of Nebraska, has accepted a position as Mathematician at the Radio Corporation of America Laboratories, Princeton, New Jersey.

Mr. Dov Avishalom, Bar Ilan University, Tel Aviv, Israel, has been appointed Instructor at the University of Minnesota.

Mr. S. D. Beck, Battelle Memorial Institute, has been appointed Nuclear Engineering Specialist at Alco Products, Schnectady, New York.

Associate Professor E. G. Begle, Yale University, has been appointed Professor of Mathematics Education at Stanford University's School of Education.

Mr. R. J. Benice, University of Buffalo, has been appointed Senior Mathematician at the Sylvania Electronic Systems, Buffalo, New York.

Mr. W. H. Benson, University of California, Berkeley, has accepted a position at the Lawrence Radiation Laboratory, Berkeley, California.

Mr. B. D. Biegun, University of Minnesota, has been appointed Teacher at Napa Senior High School, Napa, California.

Mr. J. H. Braun, Chrysler Corporation, Detroit, Michigan, has accepted a position as Systems Engineer with International Business Machines, Columbus, Ohio.

Brother H. Columban, Xavier High School, Appleton, Wisconsin, has been appointed Teacher at St. George High School, Evanston, Illinois.

Dr. S. D. Conte, Space Technology Laboratories, Los Angeles, California, has accepted a position as Manager of the Computing Sciences Department, Systems Research and Development Division, Aerospace Corporation, Los Angeles, California.

Professor F. L. Kiokemeister has been appointed Chairman of the Department of Mathematics at Mount Holyoke College.

Mr. S. J. Einhorn, University of Pennsylvania, has been appointed to the Technical Staff of Averbach Electronics as Mathematical Analyst.

Mr. Samuel Feder, Bulova Research & Development Laboratories, Woodside, New York, has accepted a position at System Development Corporation, Paramus, New Jersey.

Dr. E. H. Hanson, Land-Air, Point Mugu, California, has accepted a position as Manager of the Advanced Analysis Department, Earth Sciences Division, United ElectroDynamics, Pasadena, California.

Associate Professor M. C. Hartley, University of Illinois, has been appointed Visiting Professor at the University of Puerto Rico, Mayaguez, Puerto Rico.

Mr. N. W. Johnson, Technical Operations, Washington, D. C., has accepted a position at Computer Associate, Woburn, Massachusetts.

Professor Edgar Karst, Brigham Young University, has been appointed Professor at Evangel College.

Dr. H. C. Kennedy, St. Louis University, has been appointed Assistant Professor at Providence College.

Assistant Professor L. H. Lange has been appointed Acting Head of theDepartment of Mathematics at San Jose State College.

Professor Joseph Lehner, Michigan State University, will be on leave during the academic year of 1961–62 and will attend the Number Theory Institute at the University of Pennsylvania.

Associate Professor V. O. McBrien, College of the Holy Cross, has been appointed Professor and Chairman of the Department of Mathematics.

Mr. G. E. Murine, Head of the Mathematics Department, Solon School System, Cleveland, Ohio, has been appointed Assistant Professor at John Carroll University.

Dr. J. B. Muskat, Massachusetts Institute of Technology, has been appointed Assistant Professor and Research Associate in Computing, University of Pittsburgh.

Professor J. H. Neelley, Carnegie Institute of Technology, has been appointed Visiting Professor at Ball State Teachers College.

Mr. W. K. Rapp, University of Missouri, has accepted a position as Senior Engineer with Motorola Military Electronic Center, Scottsdale, Arizona.

Associate Professor N. C. Severo, University of Buffalo, has been promoted to Professor of Mathematical Statistics.

Mr. Lawrence Sokoloff, Sylvania Electric Products, Needham, Massachusetts, has accepted a position with Auerbach Electronics, Philadelphia, Pennsylvania.

Dr. J. W. Summers, University of California, Berkeley, has been appointed Assistant Professor at Alameda State College.

Dr. G. H. Swift, Jr., International Business Machines, Poughkeepsie, New York, has been appointed Manager, Technical Plans for Large Computers, International Business Machines, Federal Systems Division, Rockville, Maryland.

Dr. T. T. Tanimoto, International Business Machines, has been appointed Head of the Pattern Recognition Laboratory, Melpar, Watertown, Massachusetts.

Professor Emeritus W. B. Carver, Cornell University, died on July 4, 1961. He was a charter member of the Association. Professor Carver served the Association as Editor of the MONTHLY (1932–1936), President (1939–1940), and Secretary-Treasurer (1943–1947). He had the distinction of being the only person to hold these three offices. From 1947 Professor Carver continued to serve as a member of the Finance Committee until January, 1961, when he found it necessary to resign because his health was failing.

Professor E. S. Ashcraft, Stetson University, died December 17, 1960. He was a member of the Association for 10 years.

Professor Emeritus W. W. Denton, University of Arizona, died January 22, 1961. He was a Charter Member of the Association.

Assistant Professor Aaron Herschfield, Pennsylvania State University, died February 19, 1961. He was a member of the Association for 4 years.

Professor Emeritus A. J. Hoare, University of Wichita, died April 23, 1961. He was a Charter Member of the Association.

Dr. R. P. Johnson, Louisa, Virginia, died March 16, 1961. He was a member for 40 years.

Associate Professor R. B. Pinson, Stephen F. Austin State College, died February 19, 1961. He was a member of the Association for 11 years.

It has recently been pointed out that, in the notice of the death of Professor E. J. Finan, Catholic University of America (this MONTHLY, vol. 67, 1960, p. 713), the name was misspelled. The Editor regrets this error.

CONTINENTAL CLASSROOM, 1961–62

The National Broadcasting Company announced on June 28 that last season's Continental Classroom course in Contemporary Mathematics will be repeated on color tape recordings from 6 to 6:30 a.m. local time beginning September 25, 1961. Professor John L. Kelley, University of California, Berkeley, teaches *Modern Algebra* during the first semester; Professor Frederick Mosteller teaches *Probability and Statistics* during the second.

The new course on Continental Classroom will be a course in American Government This course will be televised in color and carried by approximately 170 stations in every part of the country, Monday through Friday, from 6:30 to 7:00 a.m. local time beginning September 25, 1961.

The Conference Board of the Mathematical Sciences is one of the sponsors of Contemporary Mathematics; the others are Learning Resources Institute and the National Broadcasting Company.

DOCTORAL PROGRAM FOR COLLEGE TEACHERS OF MATHEMATICS

The Graduate School of Science and the Graduate School of Education of Yeshiva University announce a new doctoral program designed primarily for college teachers of mathematics. The program is aimed toward the training of mathematicians interested primarily in college teaching, rather than in research.

The requirements of the new program include the same number of credits in graduate mathematics as is required in the Ph.D. program of the Graduate School of Science. However, the student is encouraged to select courses with an eye toward breadth of coverage, rather than intensive specialization.

In addition to the content courses, there is a sequence of courses of special pertinence to the needs of the college teacher. In particular, the course "Readings in the Masterworks of Mathematics" is designed to acquaint students with original writings in the field (using the original language when possible), and it is valuable both in itself and as a prelude to the writing of the dissertation.

For more information write to: Director of Admissions, Yeshiva University, Amsterdam Avenue & 186th Street, New York 33, New York.

OPPORTUNITIES FOR STUDY IN U.S.S.R.

The Inter-University Committee on Travel Grants, representing American colleges and universities, wishes to announce that it is soliciting inquiries and applications from graduate students and scholars who wish to spend all or part of the academic year 1962–63 engaged in study and research in the Soviet Union as participants in the academic exchange between the United States and the U.S.S.R.

American citizens under forty years of age are eligible if they are *graduate students*, *post-doctoral researchers* or *faculty members* at the time of application. A knowledge of Russian adequate to the needs of study and research is required. Other criteria for selection include intellectual ability, maturity, emotional stability, proven scholarly competance or indication of future professional promise, and substantial knowledge of both American and Russian history and culture.

Periods of study and research between one semester and fifteen months can be arranged. Funds are available to cover all or part of the exchange participant's expenses, including maintenance of family, depending on the participant's own financial needs and resources.

For further information and applications write to: Stephen Viederman, Deputy Chairman, Inter-University Committee on Travel Grants, 719 Ballantine Hall, Indiana University, Bloomington, Indiana. Applications be must received no later than December 15, 1961, to be considered for the 1962–63 exchange.

HUME MATHEMATICS HONOR GALLERY REOPENED AT MISSISSIPPI

In autumn of 1890 a young man named Alfred Hume was chosen for the Chair of Mathematics in the University of Mississippi. A few years later he was a charter member of the Mathematical Association of America and retained his membership throughout his long and active lifetime. At the very beginning of his teaching, Dr. Hume instituted a practice which turned out to be perhaps as unique as anything of its kind in the world. At the end of each year he would give critical examination of the performance of his students and honor those rated at the top by placing their pictures in the classroom. This discriminating practice resulted in groups ranging from one to perhaps seven or eight. The gallery grew and in 1924 it had gone completely around the spacious classroom and it contained names that became outstanding in many walks of life. In this year, Dr. Hume became Chancellor of the University and the practice was discontinued.

The sequel to this account is the decision made at this institution to re-activate this gallery and choose annually a recipient for the award designated as the Alfred Hume Memorial Award in Mathematics. Among the provisions of the award are membership in the Association and the privilege of having the recipient's photograph entered into this Honor Gallery which started 70 years earlier. The first recipient of this second phase is James R. Price. He will, of course, continue graduate study in mathematics in a leading mid-western University.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1961 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	Paul Eberhart, Washburn University
Missouri	R. J. Michel, Southeast Missouri State College
New Jersey	H. O. Pollak, Bell Telephone Laboratories
Northeastern	D. E. Richmond, Williams College
Ohio	R. R. Stoll, Oberlin College
Pacific Northwest	D. C. Murdoch, University of British Columbia
Southeastern	H. S. Thurston, University of Alabama
Southwestern	J. B. Giever, New Mexico State University
Upper New York State	Harriet F. Montague, University of Buffalo

The greatest number of votes (230) was cast by the members of the Northeastern Section. The highest percentage of votes was 47% in the Missouri Section.

H. M. GEHMAN, *Executive Director*

THE 1961 HIGH SCHOOL MATHEMATICS CONTEST

The Annual High School Mathematics Contest Examination, sponsored by the Mathematical Association of America and the Society of Actuaries, was administered March 9, 1961 to approximately 160,000 students in 5300 schools throughout the United States and Canada, and including 14 APO (Army) and FPO (Navy) schools, scattered widely around the world. These figures compare with approximately 152,000 students in 5200 schools in 1960.

A stronger international flavor was added to the contest examination this year by having it administered on an unofficial basis in two English schools and two Dutch schools, the latter with provision for translation. All four schools performed well, and, in communications with the teachers in the Netherlands schools, we learned that their students were very much interested in the test and that extended participation is planned for 1962. Also arrangements are being made to have the test printed in the Dutch journal *Euclides*.

On the reasonable assumption that the body of participants over the years constitute a pedagogical constant, we must conclude that this year's examination was more difficult than that of 1960. Limited statistics describing the national and regional perform-

ances are found in the summary, sent to all participating high schools and available by request from the contest chairman.

In team performance, Brooklyn Technical High School ranks first with a score of 366.75 points out of 450, and Abraham Lincoln High School (Brooklyn), second with a score of 358. The best individual performance is credited to Aviad Broshi, Yeshiva of Flatbush High School (Brooklyn) with a score of 143.75 points out of 150. Close behind is Robert Rosenstein, Abraham Lincoln High School (Brooklyn) with a score of 141.25.

The discriminatory quality of the examination on all levels of achievement continues to be satisfactory, and no changes are contemplated in basic pattern. For next year, however, we may use a dual wording, one classic and one modern, for problems based on "modern mathematics." The few sprinkled through this year's examination appeared in essentially classic garb.

To date, our most ambitious effort to interest high school students in mathematical careers is this year's three-page brochure, *How About a Career in Mathematics*, distributed to all participating students. Although the expense of printing and distributing these brochures is heavy, we are of the opinion that the educational value of the brochures warrants their continuance.

CHARLES T. SALKIND, *Chairman,*
Committee on High School Contests

THE 1961 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-second annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 2, 1961. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed about October 1 to the regular mailing list. If an application blank is not received by October 15, one may be secured by writing the director, Professor L. E. Bush, 308 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 6, 1961. For further details of the examination and the list of prizes (including the $3,000.00 scholarship to Harvard), see the announcement which will accompany the application blank.

Reports of the previous competitions and the examinations may be found in this Monthly for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947; December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; August–September 1958; August–September 1959; January (questions and solutions for eighteenth, nineteenth and twentieth competitions) 1961; and in this issue, pages 629–637.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 25, 1961, at Wayne State University, Detroit, in conjunction with the annual meeting of the Michigan Academy of Science, Arts and Letters. Professor E. D. Rainville of the University of Michigan, Chairman of the Section, presided at all sessions of the meeting. The attendance at the morning and afternoon sessions was about 80 persons and 67 attended the luncheon and business meeting.

The nominating committee consisting of Professor J. G. Hocking, Michigan State University, Chairman, Professor L. G. Woodby, Central Michigan University, and Sister M. Ignatia, Marygrove College proposed the following slate of officers: Professor

F. L. Celauro of Central Michigan University as Chairman; Professor R. H. Oehmke of Michigan State University as Vice-Chairman; Professor L. E. Mehlenbacher of the University of Detroit as Secretary-Treasurer. This slate of officers was elected unanimously.

The report of the Governor of the Michigan Section, Professor R. M. Thrall, was read by Professor Frank Harary. Professor R. K. Ritt of the University of Michigan reported for the Committee on the Michigan Mathematics Prize Competition that there were over nine thousand participants from Michigan high schools, and that the general results show improvement over previous years. Professor J. S. Frame of Michigan State University reported on the Visiting Scientist Program sponsored by the Michigan Academy and on the resolution in support of certain changes in the proposed teacher certification code for Michigan.

The morning program included the presentation of three papers and a report of the work of the Panel on Teacher Training of the CUPM. The latter report was presented by Professor R. J. Wisner, Executive Director of the CUPM. The afternoon program included the presentation of one paper and a panel discussion on the topic "The School Mathematics Study Group Programs for Elementary and Secondary School Mathematics." The Panel consisted of Professor P. S. Jones of the University of Michigan, Moderator; Miss Irene Sauble, Supervisor of Elementary Mathematics, Division of Instruction, Detroit Public Schools; Professor Charles Brumfiel, University of Michigan; Miss Hope Chipman, University of Michigan High School.

The following papers were presented:

1. *A very independent axiom system*, by Professor Frank Harary, University of Michigan.

This paper was published in this MONTHLY, vol. 68, 1961, pp. 159–162.

2. *Fitting an exponential curve to the frequencies of the lengths of precipitation periods at South Haven, Michigan*, by Professor W. D. Baten, Michigan State University.

This article contains an application of fitting an exponential curve to observed frequencies of the lengths of precipitation periods at South Haven, Michigan, for the past 30 years. The length of a precipitation period is the number of consecutive days during which there is measurable precipitation.

3. *On the set of maximum points of a regular function*, by Professor Fritz Herzog, Michigan State University.

Let F be the class of nonconstant functions $f(z)$, regular for $|z|<1$ and continuous for $|z|\leq 1$. Let F_1 be the subclass of F, consisting of those $f\in F$ which are regular for $|z|\leq 1$. The point set on the unit circle C where $|f(z)|$ assumes its maximum value will be called the *set of maximum points*. It is shown that, (a) for the class F, the most general set of maximum points is any closed nonempty subset of C, and (b) for the subclass F_1, the most general set of maximum points is either all of C or a finite nonempty subset of C.

4. *Invertible spaces*, by Professor J. G. Hocking, Michigan State University.

This paper by P. H. Doyle and J. G. Hocking is scheduled for publication in this MONTHLY, November, 1961.

L. E. MEHLENBACHER, *Secretary*

THE MARCH MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of Arizona, Tucson, Arizona, March 17–18, 1961. Professor Harvey Cohn, Chairman of the Section, presided at the afternoon session on March 17, and also at the morning session on March 18. There were 87 persons in attendance, including 40 members of the Association.

The following officers were elected: Chairman, Professor I. I. Kolodner, University

of New Mexico; Vice-Chairman, Professor Deonisie Trifan, University of Arizona; Secretary-Treasurer, Professor George Baldwin, New Mexico State University.

Dr. S. M. Ulam, Los Alamos Scientific Laboratory, gave the invited address, "Study of Combinatorial Problems on Computers."

The following papers were presented:

1. *Some computations of class numbers in quadratic integral domains*, by Professor Harvey Cohn, University of Arizona.

The author considers $h(f^2d)$ the class number for the quadratic integers of type $x+yf(d+\sqrt{d})/2$ where d is the field discriminant. Dirichlet showed that for fixed d if $f=p^aq^b\cdots$ (for a *fixed* set of primes and variable exponents), $h(f^2d)$ is bounded; also for certain sets of primes in f, $h(f^2d)=h(d)$. The survey made on GEORGE tends to suggest certain possible conjectures on the class numbers. For example: $h(12f)=1$ exactly when $f=3^a$ or $2\cdot 3^a$; $h(24f)=1$ only when $f=3$, $h(24f)=3$ only when $3<f=3^a$, and $h(24f)$ is even otherwise. When d is the sum of two squares, the values of $h(f^2d)/h(d)$ tend to be odd more often and smaller more often (as f varies) than for other d.

2. *Rings with group*, by Mr. E. L. Walter, New Mexico State University.

Let R be a ring with $1\neq 0$, and R^{**} be the set of nondivisors of zero. Define R to be a ring with group if R^{**} is a group. Several nontrivial examples are given. Ring R is a ring with group if and only if it has no proper left (right) ideals which intersect R^{**}. If the left (right) ideals of R satisfy the descending chain condition, then R is a ring with group. If X is a separable Banach space, then $B(X, X)$ is a ring with group if and only if X is finite.

3. *Closed linear operations with closed range*, by Mr. B. A. Benn, New Mexico State University' introduced by the Secretary.

Let T be a closed linear operator with domain $D(T)$ dense in Banach space X and range $R(T$ in Banach space Y. It is well known that $R(T)$ is closed in Y if and only if $R(T')$ is closed in X'. The purpose of this paper is to show that the theorem follows readily from the "state diagram." (S. Goldberg, *Linear operators and their conjugates*, Pacific J. Math., vol. 9, 1959, pp. 69–79.)

4. *A theorem on power dissipation in circuits with periodically varying elements*, by Professor I. I. Kolodner, University of New Mexico.

The response of a damped linear circuit (say in *RLC* circuit, $R>0$) to a sinusoidal electromotive force of circular frequency ω can be split into a harmonic "steady state" and a decaying transient. The Ohmic energy dissipated in the circuit over a period is a continuous function of ω on $(0, \infty)$; the energy stored in the reactive components averages to zero over a period. Under the same circumstances, if the circuit elements vary periodically with period T, the response still can be split into an almost-periodic steady state and a decaying transient. An average power P_s dissipated or stored in a set s of circuit elements can still be defined by $P_s=\lim_{t\to\infty} E_s(t)/t$, where E_s is the energy dissipated or stored in s.

THEOREM. *P_s has removable singularities on $A=\{\omega\,|\,\omega=K\pi/T,\ K\in I^+\}$, and is continuous on $(0, \infty)-A$.*

5. *Some remarks on associative functions*, by Professor Berthold Schweizer, University of Arizona.

The history of the functional equation of associativity, $F(x, F(y, z))=F(F(x, y), z)$ is discussed and a number of recent results, as well as a conjecture, concerning properties of its solutions are presented.

6. *Geometric progressions modulo 2^k*, by Mr. Milton Levy, White Sands Missile Range, New Mexico.

Elementary proofs are given of three theorems which are applicable to the generation of pseudo-random numbers in a binary digital computer. In all the following theorems, $k\geq 3$. (1) If $r\equiv\pm 1 \pmod 8$, then $r^{2^{k-3}}\equiv 1 \pmod{2^k}$. (2) If $r\equiv\pm 3 \pmod 8$, then $r^{2^{k-3}}\not\equiv 1 \pmod{2^k}$ and $r^{2^{k-2}}\equiv 1 \pmod{2^k}$. (3) If $r\equiv -3\equiv 5 \pmod 8$, then the sequence of integers generated by $x_0\equiv a$

(mod 2^k), $x_n+1 \equiv rx_n \pmod{2^k}$ for $0 \leqq n < 2^{k-2}$ is a permutation of $(1, 5, \cdots, 2^k-3)$ if $a \equiv 1 \pmod 4$ and is a permutation of $(3, 7, \cdots, 2^k-1)$ if $a \equiv 3 \pmod 4$.

7. *Differentials on a topological space*, by Professor E. D. Gaughan, New Mexico State University.

The purpose of this paper is to present the notion of a differential for a function from a topological space X into R^n, R a nondiscrete Hausdorf topological field, and to show that this differential possesses some of the properties of the usual differential as defined on a vector space.

8. *Some remarks about moment sequences*, by Professor J. W. P. Mayer-Kalkschmidt, University of New Mexico.

The following theorems are proved: (1) Given the moment sequence $u_n = \int_0^1 t^n dg(t)$, where tot. $\text{var.}_{[0,1]}(g) = K$, if $\sum_{\nu=0}^{\infty} |a_\nu| K^\nu < \infty$, then $\delta_n = \lim_{k\to\infty} \sum_{\nu=0}^{k} a_\nu \mu_n^\nu = \delta_n^k$ is a moment sequence, and $\{\delta_n^{(k)}\}$ converges to $\{\delta_n\}$ in the norm topology. (2) Under the conditions of (1), the Hausdorff method (H, δ) is regular if (H, μ) is regular and if $\sum_{\nu=0}^{\infty} a_\nu = 1$, where the a_ν are real. The case: $\mu_n = 1/(n+1)$, $a_\nu = \alpha^\nu$ is studied for $1 \leqq \alpha$, and $\alpha < 1$.

9. *General block decompositions of a space and the general computation of homology groups*, by Professor J. B. Giever, New Mexico State University.

Although the homology groups of, for example, a polytope are usually defined using simplicial decompositions, they are usually computed using more general "block decompositions" of one type or another. By using a much greater generality in the type of blocks, one can have the homology groups of the blocks determine that of the space without having a very effective means of computing by means of the blocks. Nevertheless, such a procedure can be shown to have quite useful applications.

10. *Discriminants and determinants*, by Professor Gordon Pall (Illinois Institute of Technology), Visiting Professor, University of Arizona.

The theory of quadratic forms with coefficients in a ring R in which division by 2 is either impossible or not unique has been largely neglected, principally because such forms lack symmetric matrices and determinants. By a simple device, it is shown how to associate with a quadratic form having coefficients in any commutative ring a polynomial in its coefficients, called the discriminant, which behaves much like the usual determinant and makes possible extensions of the classical theories. If R is not commutative, a similar method is given for Hermitian quadratic forms.

11. *A condition for convexity in non-Euclidean geometry*, by Professor John Irwin, New Mexico State University.

Poincare's hyperbolic model of non-Euclidean geometry consists of the points in the interior Ω of the unit circle. The lines in this geometry are arcs of circles orthogonal to the circumference of the unit circle. A set $S \subset \Omega$ is convex if and only if for each pair of points w_1 and w_2 in S, the (non-Euclidean) line segment joining w_1 and w_2 is contained in S. A condition for the convexity of regions in the complex plane is given and it is shown that a corresponding theorem for Ω holds.

12. *Some unsolved problems concerning extensions of Abelian groups*, by Professor E. A. Walker, New Mexico State University.

This paper is a discussion of some of the unsolved problems concerning $\text{Ext}(A, B)$. Recent progress on some of these problems is indicated, including some recent theorems of the author. Whitehead's problem concerning $\text{Ext}(A, Z)$ when Z is the additive group of integers is given particular attention.

13. *Set functions and their determining collections*, by Dr. H. H. Wicke, Sandia Corporation, Albuquerque, New Mexico.

It is shown (as a special case of a more general theorem) that if M is a set, then every function whose domain and range is the power set of M corresponds to a mapping of M into collections of subsets of M. Certain interesting properties of set functions such as isotonicity are characterized by properties of the corresponding collections. The assignment of neighborhoods to each point of

M and the determination of a topology is an application. Another application is a characterization of the class of all isotone set functions f satisfying $f=cfc$, where c is the complement function.

14. *Banach spaces isomorphic to a conjugate space*, by Professor Seymour Goldberg, New Mexico State University.

Let X be a Banach space. New and simple proofs are given of theorems concerning necessary and sufficient conditions that X be linearly homeomorphic to a conjugate space of a Banach space.

15. *Some issues in mathematics education today*, by Professor Charles Wexler, Arizona State University.

A report on the Board of Governors' meeting, the CUPM meeting, and some of the problems in Mathematics Education discussed by these groups.

16. *Discussion*: *Engineering curriculum in mathematics*, Chairman: Professor J. W. P. Mayer-Kalkschmidt, University of New Mexico.

17. *Discussion*: *Undergraduate program in mathematics*, Chairman: Dr. J. R. Foote, University of New Mexico and Holloman Air Force Base.

G. L. BALDWIN, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The 48th regular meeting of the Iowa Section of the Mathematical Association of America was held at Simpson College, Indianola, on the afternoon of April 14 and the morning of April 15, 1961. Professor Irvin Brune presided as Acting Chairman in behalf of Professor H. C. Trimble, Chairman of the Section, who could not be present. Total attendance was 81, including 35 members of the Association. Routine business was considered during the afternoon meeting of April 14.

The following officers were elected: Chairman, Professor Hazel M. Rothlisberger, University of Dubuque; Vice-Chairman, Professor W. L. Waltmann, Wartburg College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program.

1. *Proof of the remainder theorem*, by Professor H. A. Heckart, Simpson College.

2. *Integration of* $\cot x \sin 2mx \ln(\sin x/\sin a)dx$, by Professor Don Kirkham, Iowa State University.

For $m=1, 2, \cdots$, the title integral and some related ones are evaluated. The results are obtained in terms of psi-functions.

3. *A mixed boundary value problem for an infinite elastic cylinder*, by Professor Harry Weiss, Iowa State University.

4. *A thin cylindrical shell problem*, by Mr. Thomas Rogge, Iowa State University, introduced by the Acting Chairman.

The problem of a thin cylindrical shell sector clamped on two curved edges and one straight edge, free on the remaining straight edge and loaded by a load similar to a hydrostatic load is considered. The method of solution is that of superimposing the solutions of three separate problems with the appropriate boundary conditions.

5. *Number theoretic densities for the Gaussian integers*, by Mr. W. H. Richardson, Iowa State University.

Some of the standard number-theoretic densities are defined for the Gaussian integers; and for these are established results analogous to corresponding cases for the rational integers. For example, one possible asymptotic density is given by $\liminf A(n)/N(n)$, where $A(n)$ is the number of elements of A in the half-open square determined by 0, n, and ni, and $N(n)$ is the norm of the Gaussian integer n.

6. *A modified Runge-Kutta solution of ordinary differential equations*, by Mr. G. D. Byrne, Cyclone Computer Laboratory and Professor R. J. Lambert, both of Iowa State University.

Suppose it is required that a particular solution to the differential equation $dy/dx = F(x, y)$ be found. Suppose further that the points (x_{n-1}, y_{n-1}) and (x_n, y_n) lie on the particular solution curve and that they are given. Let $x_n + h = x_{n+1}$, $y_n + k = y(x_{n+1}) = y_{n+1}$. Here h, the step-size, is fixed. Therefore, k, the change in y, must be evaluated to find the next point, (x_{n+1}, y_{n+1}), on the particular solution curve. A set of equations is given describing a modified Runge-Kutta method of numerical integration, which has an accuracy of the order of h^3 and which requires only two substitutions into the differential equation for each step of integration.

7. *A note on Boolean algebras*, by Professor M. F. Ruchte, Iowa State University.

8. *Integral transforms and boundary value problems*, by Mr. Gary Anderson, Iowa State University, introduced by the Acting Chairman.

9. *Lilliputian dynamics—the physics of extreme size change*, by Mr. Robert Gordon, Bettendorf High School, introduced by the Acting Chairman. (By invitation)

Whenever the size of an object or animal is changed, the scale factor must be considered. It was discovered experimentally that because strength is proportional to cross-sectional area, while mass is proportional to volume, a large animal or object, built similar to a smaller one, will be weaker, proportionately, by a factor of scale. Also, the smaller an object or animal becomes, the more surface area, relative to mass, it has. This explains why very small animals seem extremely hungry and are easily water-logged. Scaling is also important when the behaviorial properties of various size ships are considered. These are but a few of the many aspects of scaling.

10. *Recommendations of CUPM Panel on Teacher Training*, by Professor W. R. Orton, University of Arkansas.

11. *Iowa college reactions to CUPM recommendations*, by staff members of representative Iowa Colleges: Professor J. O. Chellevold and Professor O. C. Kreider, Iowa State University, Mr. N. L. Jacobson, Graceland College, Professor H. V. Price, State University of Iowa. Questions and comments from the floor.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-sixth annual meeting of the Kansas Section of the Mathematical Association of America was held at Ottawa University, Ottawa, Kansas, on April 15, 1961, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. Of the 195 people registered, 72 were members of the Association. Professor W. D. Bemmels, Chairman, presided at the sessions.

The following officers were elected for one-year terms: Chairman, Professor L. E. Fuller, Kansas State University; Vice-Chairman, Professor A. M. Wedel, Bethel College; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State College.

At the joint session, held in the morning, Professor J. G. Kemeny, Dartmouth College, addressed the group on *Recommendations for the Training of Teachers of Mathematics*.

The following short papers were presented at the afternoon session:

1. *Decision procedures for a propositional calculus*, by Professor B. J. Thorne, Kansas State University.

The calculus of material implication has the remarkable property of being decidable. The decision procedure in this case is the well-known system of truth tables. This paper presents a generalization of this method. An abstract definition of "propositional calculus" is given. "Finite model of a propositional calculus" is then defined and shown to have some very useful properties as a decision procedure for propositional calculi in general.

2. *Theorems on congruence*, by Professor A. A. Nafoosi, Kansas State College.

THEOREM 1. *Every odd number is the sum of at most nine odd squares, and every even number is the sum of at most ten odd squares. The proof depends on Gauss' theorem that every number is the sum of three triangular numbers.* THEOREM 2: *If* $s \geqq 4$, $N = s \pmod{B}$ *and* $h \geqq 1$, *then* $\sum_{i=1}^{s} (Bx_i+1)^2 \equiv N \pmod{p^h}$ *is always solvable for* $p \neq 2$. The proof is carried out by induction on h.

3. *Axiomatic set theory*, by Professor John Johnston, University of Kansas.

Axiomatic set theory is considered to be a portion of a general first order predicate calculus whose object language contains only bound formulae. The primitive symbols are indicated and the concept of formula defined in such a manner that no individual variable lies within the scope of two quantifiers acting on that variable. Rosser's six axiom schemata and rule of inference are stated in a manner that produces only bound formulae. After machinery for logical manipulation is developed, axioms governing the binary predicate constant "ϵ" are stated, essentially as in Gödel.

4. *Orthogonality conditions for polynomial solutions of a class of fourth order linear differential equations*, by Professor J. W. Meux, Kansas State University.

A class of fourth-order linear differential equations, each equation having a single set of polynomial solutions, is considered. A set of five conditions, sufficient to insure orthogonality of the solution set with respect to a weight function, over the fundamental interval (a, b), is derived. Under these conditions, analogues of the classical orthogonal polynomials, as well as other sets, may be obtained as solutions of this class of fourth-order differential equations.

5. *Lipschitzian homeomorphisms of bounded convex sets*, by Professor Robert Adams, University of Kansas.

Let R_1 and R_2 be bounded open convex sets in a normed linear space H, and $\rho_1(x)$, $\rho_2(x)$ their respective support functions. It is shown that the mapping $Tx = [\rho_1(x)/\rho_2(x)]x$ is a Lipschitzian homeomorphism of H onto H such that $T(R_1) = R_2$. The Lipschitzian constant (for T and T^{-1}) is bounded by $(1+r_2/r_1)^2$, where r_1 and r_2 are the radii of a pair of concentric spheres contained in and containing, respectively, both R_1 and R_2.

HELEN F. KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Annual Spring Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Aberdeen Proving Ground, Aberdeen, Maryland, on Saturday, April 29, 1961. Professor D. B. Lloyd, Chairman of the Section, presided. Sixty-three persons were present, including 60 members of the Association.

The following officers were elected to serve during the year 1961–62; Professor W. K. Morrill, Johns Hopkins University, Chairman; Professor J. C. Abbott, U. S. Naval Academy, and Mr. L. K. Meals, David Taylor Model Basin, Vice-Chairmen; Professor S. B. Jackson, University of Maryland, Treasurer, and Professor Herta T. Freitag, Hollins College, Secretary. Professor D. B. Lloyd and Dr. Michael Goldberg will represent the Section on the Joint Board on Science Education of the Washington, D. C. area.

The following papers were presented:

1. *A new characterization of Boolean algebras*, by Professor J. C. Abbott and Mr. P. R. Kleindorfer, U. S. Naval Academy, presented by Mr. Kleindorfer.

Since most mathematical statements are in the form of implications, a system of logic based on such a concept may be analytically valuable. This paper defines an implication algebra by an existence and closure postulate and five axioms. This unrestricted implication is equivalent to con-

ventional "material implication." Defining a "contained-in" operation for implication algebras, "formal implication" results, and partial ordering under this operation is established. Join and meet operations are defined and discussed, distributive and absorptive laws are proved. Upon introduction of a null element, implication algebras are shown to be Boolean algebras. Theorems and illustrations are given.

2. *On the converging factor for the asymptotic series connected with the exponential integral*, by Dr. F. D. Murnaghan, and Dr. J. W. Wrench, Jr., Applied Mathematics Laboratory, David Taylor Model Basin, Washington, D. C.

The converging factor for the asymptotic series $1+1!/x+2!/x^2+\cdots$ is discussed. When x is negative, the leading term of the series which furnishes the converging factor is $\frac{1}{2}$, while when x is positive, the leading term is $\frac{3}{4}$. The problem of determining the converging factor is more difficult in the latter case than in the former, and heretofore only four terms of the series which furnishes it have been given. The present paper gives twenty-one terms of this series, and these enable us to calculate, for example, $Ei(10)$ correct to within a unit in the twenty-third decimal place. As a by-product, we obtain twenty-one terms of the Stirling asymptotic series for $\Gamma(x)$.

3. *Two-dimensional slide rule—reduced cubic equation*, by Mr. C. R. White, Ballistic Research Laboratories, Aberdeen Proving Grounds, Maryland.

This slide rule is designed to solve not only the reduced cubic equation, $w^3+3pw+2q=0$, but also the general quadratic equation, $x^2+2ax+b=0$. It gives simultaneously for either equation; (1) the roots real and/or complex with no ambiguity in signs, (2) the complex roots in both rectangular and polar forms, (3) all numerical answers on ten-inch logarithmic scales. These results make possible approximate solutions to the general cubic and quartic equations and thus lead to direct computations for root-gain loci problems corresponding to transfer functions of the types $G_{n,m}(s)$ for $n=2, 3, 4$ and $m=1, 2, \cdots, n-1$.

4. *A procedure for determining a family of minimum-cost network flow patterns*, by Mr. R. G. Busacker, Johns Hopkins University.

Given a finite network (linear graph) joining nodes A and B, with a "unit cost" (nonnegative real number) and "capacity" (nonnegative integer) associated with each directed link, a procedure is presented for solving jointly the following set of linear programming problems: For every positive integer k, produce (when one exists) a feasible flow pattern from A to B minimizing total cost subject to "delivering" k units. A feasible flow pattern delivering k units from A to B is an assignment of "link flows" (integers) which do not exceed the corresponding link capacities, such that output equals input at nodes other than A and B and output minus input equals k at A.

5. *Asymptotic behavior in a problem of subsidence*, by Mr. J. H. Giese, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

T. Leser and A. Jenike have proposed a mathematical model of the subsidence of material in overstope mining that can be reduced to the consideration of $v(t)=(1+t)^{-2}+a\int_0^t(1+r)^{-2}v(t-r)dr$. Here $v(t)$ is the velocity of the disturbance wave advancing upward into undisturbed overburden, t is time, and a is a positive constant. By elementary methods it can be proved that as t becomes infinite $v(t)$ approaches infinity or zero accordingly as a is greater or less than unity. In the latter case the disturbance rises only a finite distance. Similar results can be obtained by Laplace transforms.

6. *On relation of distribution functions to the one-step functions*, by Dr. Ceslovas Masaitis, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Let M be a normed linear space of real valued bounded functions defined on the space T of reals with the norm $\|x\|=\sup_{t\in T}|x(t)|$. Let K be the subset of M consisting of all functions K of the type $K(p, t)=0$ if $t<p$, 1 if $t\geq p$. Then the set D of all distribution functions is identical with the closure of the convex hull of K.

HERTA T. FREITAG, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held on April 22, 1961, at the University of Missouri, Columbia, in conjunction with the meeting of the Missouri Council of Teachers of Mathematics. Professor J. L. Zemmer, Chairman of the Section, presided at the morning session and Professor J. J. Andrews, Vice-Chairman of the Section, presided at the afternoon session. A total of 89 persons attended the meetings, including 39 members of the Association.

The officers elected for 1961–62 are: Professor R. M. Rankin, Missouri School of Mines as Chairman, Professor C. V. Fronabarger, Southwest Missouri State College as Vice-Chairman, and Professor C. A. Johnson, Missouri School of Mines as Secretary-Treasurer. A report of the Annual High School Mathematics Contest sponsored by the Section was given. A motion that the Section continue to administer the advanced placement examination in algebra and trigonometry for another year was passed.

The following papers were presented at the morning session:

1. *Multiplicative functions with special reference to Ramanujan's trigonometrical function* $C_m(n)$, by Professor M. V. S. Rao, Department of Mathematics, Sri Venkateswara University, Triupati, India; Visiting Professor, University of Missouri.

Using the fact that multiplicative arithmetic functions of r arguments form an abelian group under suitable operations like composition and compounding, various properties of these functions have been derived. This approach has been used to study Ramanujan's trigonometric function (representing the sum of the nth powers of the mth order primitive roots of unity) and its various generalizations, together with their applications to various problems involving relative partitions.

2. *The Advanced Placement Tests for Missouri high school seniors*, by Professor J. J. Andrews, St. Louis University.

The Misouri Section of the M.A.A. is sponsoring a state wide advanced placement examination in algebra and trigonometry. Successful completion of this examination qualifies a student to enter analytic geometry and calculus when enrolling at a cooperating university in Missouri. College credit is decided by the individual university. Thirty-six universities and junior colleges are participating in the program, where 998 students from 153 high schools have signed up for the examination. The concern of the high schools and the interest of the students is most encouraging.

At the joint afternoon session Professor W. R. Orton, Jr., University of Arkansas, addressed the group on *Recommendations for the training of teachers of mathematics*. This was a discussion of the report of CUPM.

NOLA A. HAYNES, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-seventh annual meeting of the Nebraska Section of the Mathematica Association of America was held on April 14 and 15, 1961, at the University of Nebraska, Lincoln, Nebraska, in conjunction with the seventy-first annual meeting of the Nebraska Academy of Sciences. Professor R. L. Moenter, Chairman of the Section, presided. There were 75 persons present at the sessions, including 30 members of the Association.

The following officers were elected for 1961–62: Chairman, Professor W. E. Mientka, University of Nebraska; Vice-Chairman, Professor R. L. Moenter, Midland College; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska. Professor J. M. Earl was continued as Chairman of the Committee on Mathematics Contests. The Committee consists of representatives of the Nebraska Section of the Mathematical Association of America, the Nebraska Section of the National Council of Teachers of Mathematics, the Nebraska Actuaries Club, and the Nebraska Academy of Sciences.

The following papers were presented:

1. *Properties of solutions of* $u''+g(t)u^{2n-1}=0$, by Professor W. R. Utz, University of Missouri. (By invitation)

Let $g(t)$, defined for all real t, be an oscillation coefficient provided there exists a real number K such that for all k, $K>k>0$, the solutions of $u''+kg(t)u=0$ are oscillatory. Several known theorems give sufficient conditions that $g(t)$ be an oscillation coefficient. The principal theorem of the paper asserts that if $x=x(t)$, $x(t)\not\equiv 0$, is a solution of $x''+g(t)x^{2n-1}=0$ valid for all large t and if $g(t)$ is continuous, $g(t)>0$ and $g(t)$ is an oscillation coefficient, then $x(t)$ oscillates as $t\to\infty$.

2. *Bases and infinite series in Banach spaces*, by Professor Gerald Heuer, Visiting Assistant Professor, University of Nebraska. (By invitation)

Several definitions of bases for infinite dimensional topological vector spaces are possible. A Hamel basis always exists (and does not require a topology). In a Banach space E, the following has been considered by several authors. A sequence $\{x_n\}$ in E is a B-basis if for every x in E, there exists a unique set of scalars $\{a_n(x)\}$ such that $\sum_{n=1}^{\infty} a_n(x)=x$. If the convergence is unconditional, it is an absolute B-basis. Existence of a B-basis requires E to be separable. Whether separability implies existence of a B-basis is unknown.

3. *Some theorems and problems in combinatorial geometry*, by Professor W. R. Utz, University of Missouri. (By invitation)

Beginning with a 1935 paper of Vincensini, suggested by the well-known Helly theorem for convex sets, the speaker considered the problem of restricting families of plane sets, generally convex, in such a way that one can determine a positive integer n such that if each n of the sets are intersected by a line, then all sets of the family are intersected by some line. Attention was confined to finite families and results of Santalo, Hadwiger, Danzer, and Grünbaum were emphasized in which the families are parallelograms, line segments, or circles.

4. *The Fourth Nebraska (Twelfth National) Mathematics Contest*, by Professors J. M. Earl, University of Omaha, and H. M. Cox, University of Nebraska, presented by Professor Earl.

A total of 242 schools (31 new this year) have participated in one or more of the four Nebraska Contests. There have been 1635, 2428, 2616, and 2828 students from 127, 133, 140, and 140 schools in the first, second, third, and fourth contests, respectively. Some 580 students participated in both the third and fourth contests; their median scores were 11 and 13, respectively ($r=0.53$). The name of one Nebraska contestant appears on the National Honor Roll.

5. *A physical study of hail*, by Professor L. V. Andrews, Nebraska State Teachers College, Chadron, introduced by the secretary.

Measurements of the temperature of rainfall, nucleating properties of silver iodide as determined by cooling curves, silver iodide concentration in precipitation, and physical characteristics of hail have been made in an effort to distinguish between thunder storms which have been seeded with silver iodide and those which proceed with natural nucleating agents.

6. *Recommendations for the training of teachers of mathematics*, by Professor Henry Van Engen, University of Wisconsin. (By invitation)

7. *The Computing Center of the University of Nebraska*, by Professor John Christopher, University of Nebraska.

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE SOUTHEASTERN SECTION

The fortieth annual meeting of the Southeastern Section of the Mathematical Association of America was held April 7–8, 1961, at Wofford College, Spartanburg, South Carolina. Professors E. B. Shanks, Vice-Chairman of the Section, D. E. South, J. W. Lasley, J. V. Hancock, and Nathaniel Macon presided over the general and divisional sessions. There were 139 members and guests of the Association in attendance.

The following officers were elected for the coming year: Chairman, Professor E. B.

Shanks, Vanderbilt University; Vice-Chairman, Professor Anne L. Lewis, Woman's College, University of North Carolina. A motion was passed that the term of office of the Secretary-Treasurer be changed from one to three years and that no holder of this office be re-elected for more than one term. Professor C. L. Seebeck, Jr. was elected to this position for the first three-year term. Professor R. M. Thrall, University of Michigan and Visiting Lecturer for the MAA, was the featured lecturer at the meeting. A motion that the Section maintain its present boundaries was approved. The invitations of the Woman's College of North Carolina for the 1962 meeting and of the University of Chattanooga for the 1963 meeting were confirmed and of the Citadel for 1964 and of North Carolina State University for 1965 were accepted.

The following program was presented:

1. *Set-valued classes and the axiom of exchange in set theory*, by Professor E. B. Shanks, Vanderbilt University.

A class A is set-valued if and only if for each set v there exists a set x such that for each set u, if the ordered pair $\langle uv \rangle$ belongs to A then u belongs to x. The Exchange Axiom is then stated: $(\alpha, A)\{\text{Set-val}\,(A).\rightarrow.(\exists y)(u)[u\in y.\equiv.(\exists v)(v\in\alpha.\langle uv\rangle\in A)]\}$. In this statement, α is an ordinal set. If α is replaced by a general set x, the axiom is called the Generalized Exchange Axiom, which is provable in the author's axiom system. In Godel's system, it will replace the sum and substitution axioms.

2. *On polynomial elimination theory*, by Professor L. H. Williams, Duke University.

Polynomial elimination theory is often eliminated from senior and graduate courses in algebra probably for the reason that use of the theory is not practical for pencil and paper calculations. The IBM 650 computer has been taught to do the necessary formal manipulations on polynomials in several variables, thus making classical elimination theory practical. The theory and applications are discussed.

3. *A method for the computation of the greatest root of a positive matrix*, by Professor Alfred Brauer, University of North Carolina.

4. *Computer short circuits*, by Professor W. G. Miller, Clemson College.

The title refers to circuitry which may simplify certain computer operations. Developments stem from "human" reasoning. (1) Extraordinary advances resulting from intuition and flashes of inspiration in processes of human thinking overshadow the fact that much of the reasoning involves merely the elimination of conditions conflicting with explicitly defined rules. This is translated into circuits which include more sophisticated binary concepts such as IF-THEN, IF-AND-ONLY-IF, OR-ELSE, etc. (2) The capacity of the human mind for manipulating "weighted" inputs is interpreted by two "multi-valued" logic techniques: (i) combinations of analogue-type inputs (potentiometers) with digital-type operations; and (ii) use of ac-current and transformer-flux relations.

5. *The use of personal equations in teaching numerical analysis*, by Professor E. P. Miles, Jr., Florida State University.

When learning to find real roots of polynomial equations by the Newton's and False Position methods, each student forms a polynomial with coefficients determined from his birth date and year. Although the polynomials formed differ for each student, all have a zero between 0 and 1. Students locate the smallest such zero to successive tenths by a preliminary analysis and then program the IBM 650 to approximate this root to six decimal places using the two methods studied. The machine thus does most of the grading which would otherwise be quite tedious with each student doing a different problem.

6. *On the existence of a nonzero form*, by Professor E. H. Hadlock, University of Florida.

The purpose of this paper is to show the existence of a nonzero ternary quadratic form $f=ax^2+by^2+cz^2+2ryz+2sxz$ with arbitrary a and of determinant d, when the invariants Ω and Δ of f are given. Formulas or conditions involving Legendre's and Jacobi's symbols were derived which

when satisfied by a given set of conditions show the existence of a form f. Next formulas compatible with the previously mentioned formulas were derived and when satisfied show the existence of a nonzero indefinite form f. These formulas are readily applicable to the construction of a positive, or a nonzero indefinite form f.

7. *Linear fractional Tschirnhaus transformations in algebraic fields*, by Professor H. S. Thurston, University of Alabama.

If ρ is a root of an irreducible equation of degree n over a field F, and η is a primitive number of $F(\rho)$, necessary and sufficient conditions are found such that ρ shall be expressible in the form $(a\eta+b)/(c\eta+d)$ where a, b, c, and d are in F.

8. *On ternary rings*, by Professor J. R. Wesson, Vanderbilt University.

The definition of a ternary ring requires that there exists an element e such that $ea0=ae0=a$ for all a. Let S be the system obtained by dropping the postulate above. Then let xyz indicate the ternary operation in S, and define a new ternary operation $\langle xyz\rangle$ on the elements of S by selecting $e\neq 0$ and defining $\langle xyz\rangle = xaz$, where $ea0=y$. The new system satisfies the conditions required of S, and furthermore $\langle ey0\rangle = y$ for all y. Similarly, a system S satisfying $ey0=y$ for all y can be transformed into a system satisfying $\langle ey0\rangle = \langle ye0\rangle = y$ for all y.

9. *A method for constructing involutory matrices*, by Professors Jack Levine and H. M. Nahikian, North Carolina State College, presented by Professor Nahikian.

A method for constructing all involutory matrices over a field $\mathfrak{F}$ is developed in this paper. The essential theorem proven may be stated: A necessary and sufficient condition that an $n\times n$ matrix H with elements in a field $\mathfrak{F}$ be involutory is that there exist column vectors X_α, $Y_\alpha \in V_n(\mathfrak{F})$ $(\alpha = 1, \cdots, s,\ s \leqq [\tfrac{1}{2}n])$, such that $H = I + \sum_{\alpha=1}^{s} Z_\alpha$, $Z_\alpha = X_\alpha Y_\alpha'$, where $X_\alpha' Y_\beta = -2\delta_\alpha^\beta (\alpha, \beta = 1, \cdots, s)$.

10. *On diophantine equations of the form* $x^n+y^n=hp^m$, by Mr. J. E. Shockley, University of North Carolina.

11. *Stress distribution of a rotating limaçon*, by Mr. J. L. Tilley, University of Florida.

Consider a thin plate of isotropic material in the shape of a limaçon given by the equation $r=2\alpha+2\beta\cos\theta$, $\alpha>\beta$, rotating in its plane about an axis through its centroid. The method of attack is to consider the equivalent problem of the limaçon in a fixed position acted on by appropriate body forces and is further simplified by considering the body forces in the x-direction and the y-directed separately. The final solution is thus found by superposition of these two results.

12. *The stress distribution due to triangular discontinuities in plates under edge forces*, by Professor C. B. Smith, University of Florida.

A large rectangular plate lying in the xy-plane is subjected to a uniform tension in the y-direction. If the plate is homogeneous, the stress distribution is quite simple. However, if a small rigid triangular region is assumed to lie in the center of the plate, the stress distribution is, of course, considerably changed. The stress distribution near various triangular shaped regions are discussed and the results contrasted with the stresses arising when open holes of similar shape occur in the plate.

13. *On particular values of the derivatives of the gamma function*, by Professor R. W. Cowan, University of Florida.

By successively differentiating the relation $\Gamma(x+1)=x\Gamma(x)$, employing the duplication and other formulas from the theory of the gamma function, values are obtained for the first two derivatives of the gamma function for a positive integer and half of an odd positive integer. These results are established by mathematical induction.

14. *A method to obtain conformal mapping functions by a direct approach and its application*, by Professor S. F. Yeung, University of Florida.

By means of a direct consideration of the polar form of the boundary of a certain type of

simply connected domain D, a conformal mapping which will map the unit circle onto domain D can be obtained. The mappings thus obtained are applied to solve torsion problems in elasticity.

15. *Wifeless tournaments for mixed doubles*, by Professor C. W. Huff, Winthrop College.

This paper supplements a couple of recent papers in this MONTHLY on designing mixed doubles tournaments by giving an arrangement of a schedule of play for couples' party bridge. The restrictions are that there shall be at least six matches (or rounds), and no two players shall play at the same table more than once. An additional restriction is easily supplied; namely, that husband and wife shall not play at the same table any time during the tournament.

16. *Comments on honors courses*, by Professor P. K. Smith, University of South Carolina.

The approach to the question of honors courses was that of setting forth the major inherent difficulties involved with the intent of evoking statements of views that might be helpful to those present with an interest in the subject of the paper. The different plans for challenging the superior student were outlined. The problem of setting up classes for superior students, as well as a general honors program, for a university was discussed. The problem of selecting the students to be invited into classes for superior students and the selection of the most capable teachers for these classes was considered. The need of reducing the load of the teacher handling honors classes was especially emphasized.

17. *A discussion of some of the steps taken by the Department of Mathematics at North Carolina State College to improve the mathematical competence and achievement of its students*, by Professor H. V. Park, North Carolina State College.

In this paper specific attempts to improve the mathematical competence and achievement of students in two areas is discussed. First, significant progress has been made, relative to high school students and students from junior colleges planning to transfer to North Carolina State, through the following programs: (a) participating in NSF sponsored institutes for high school mathematics teachers, (b) conducting a special summer program for a select group of rising high school seniors, and (c) participating in a joint university-small college conference. Secondly, capable students, upon entering State College, are given an opportunity to advance academically and to pursue more depth in mathematics by (a) being certified for credit by examination, and (b) participating in the Superior Student Program in Mathematics.

18. *Are sets omnipotent?* by Professor C. G. Phipps, Tennessee Polytechnic Institute.

Carried away by their enthusiasm, the proponents of "modern mathematics" are claiming too much for it. In many cases the word *modern* could be replaced by the word *fashionable*. Set theory is an example. The axiomatic approach to such theories is similar to the postulational approach to geometry of past years now discarded as pedagogically harmful. Another questionable practice is the inclusion of assumptions in proposed definitions.

19. *Plane geometry and complex numbers*, by Professor R. G. Blake, University of Florida.

The complex number z and its complex conjugate $\bar{z}$ can be used instead of x and y in the equations of plane figures. Such equations are useful in studying the effect of conformal maps from the z-plane to the w-plane. Equations are developed for the straight line and the conic sections.

20. *A note on Baire functions of the first class*, by Professor R. D. McWilliams, Florida State University.

If a bounded Baire function f is the pointwise limit of a uniformly bounded sequence $\{f_n\}$ of continuous real functions on a closed interval, then f is the pointwise limit of a sequence $\{g_n\}$ of finite linear combinations of $f_1, f_2, \cdots$ such that for each n the least upper bound of $|g_n(t)|$ on the interval is the same as that of $|f(t)|$. If the sequence $\{f_n\}$ is not unformly bounded, the conclusion need not be true.

21. *Exterior products in analytic geometry*, by Professor Johann Sonner, University of South Carolina.

In elementary vector algebra an oriented area (resp. volume) is described by a product $x\Lambda y$ (resp. $x\Lambda y\Lambda z$) which unfortunately is identified with an element of R^2 (resp. R). One should not make this identification, but regard $x\Lambda y$ (resp. $x\Lambda y\Lambda z$) as a new entity called bivector (resp. trivector). Let E be a vector space over R of finite dimension n. Extending the previous method an oriented p-dimensional volume in E may be described by a product $x_1\Lambda \cdots \Lambda x_k$. This suggests the construction of an exterior algebra $\wedge E$ whose elements are linear combinations of multivectors $x_1\Lambda \cdots \Lambda x_p$ and obey the rules: $x\Lambda(y+z)=x\Lambda y+x\Lambda z$; $x\Lambda x=0$. Exterior algebras are useful in finding determinants, volumes, distances.

22. *Groups having every subgroup as a direct summand*, by Miss Wai-Kit Leung, University of South Carolina.

If a group G is finite and every subgroup is a direct summand, then G is the direct sum of cyclic subgroups each having order some prime, and the converse is true.

23. *A probability distribution function*, by Professor D. E. South, University of Florida.

For the problem of sampling, without replacement, from a finite population, a general expression is obtained for the probability density function of x successes. Defining $P_{a,b}(n, x)$ as the probability of x successes in n trials, beginning with the ath trial, having had b successes, and α_{ij} as a mapping function from $P_{a,b}(n, x)$ to $P_{a+i,b+j}(n, x)$, the density function for $P_{1,0}(n, x)$ is shown to be $P_{1,0}(n, x)=\sum_\tau(p_{1,0}\alpha_{1,1})^x(q_{1,0}\alpha_{1,0})^{n-x}P_{1,0}(0, 0)$. The summation extends over the products of all possible permutations of n operator factors, x of which are $p_{1,0}\alpha_{1,1}$ and $n-x$ are $q_{1,0}\alpha_{1,0}$.

C. L. SEEBECK, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the Mathematical Association of America was held at the Stephen F. Austin State College, Nacogdoches, Texas, on April 14–15, 1961. There were 175 persons present, including 122 members of the Association. Professor H. S. Vandiver of the University of Texas was the invited speaker. Officers for the next year are: Chairman, Professor W. I. Layton, Stephen F. Austin State College; Vice-Chairman, Professor G. R. MacLane, Rice University; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The following papers were presented Friday afternoon and Saturday morning:

1. *A lemma for maximum and minimum values of functions of many variables*, by Professor H. A. Luther, Agricultural and Mechanical College of Texas.

Let the function to be studied be $f(x_1, \cdots, x_n)$. Let $n-r$ of the first-order partial derivatives when equated to zero determine uniquely the functions $x_k=g_k(x_1, \cdots, x_r)$, where $k=r+1, \cdots, n$. Then under suitable circumstances one may study instead $f(x_1, \cdots, x_r, g_{r+1}, \cdots, g_n)$.

2. *Invariants in extended analytic geometry*, by Professor R. S. Underwood, Texas Technological College.

A locus on the XY-plane for an equation in n variables, as obtained by the rules of extended analytic geometry, is called a *silhouette* if any part of the plane, with the exception of lines and points, remains uncovered. It has been proved for various general cases that all silhouettes of any given quadratic equation, as obtained from different linear plotting rules, are invariant in a topological sense. That is, one such silhouette infallibly shows the nature of all others. Furthermore, the silhouette reveals certain intrinsic algebraic properties of the equation.

3. *Wiener's smoothing and prediction technique as an extension of the least squares technique*, by Professor E. R. Keown, Agricultural and Mechanical College of Texas.

This paper is a heuristic discussion of Wiener's root-mean-square technique of prediction and filtering. The integral equation of the theory is replaced by an infinite series and the observation made that for stationary time series the technique becomes that old time favorite "the method of least squares."

4. *On developments in an arithmetic theory of the Bernoulli numbers*, by Professor H. S. Vandiver, University of Texas.

Consider the recursion formula $(b+1)^k = b_k$ for $k>1$. If the left-hand member of this equation is expanded by the use of the binomial theorem, and b_i is substituted for b^i, $i=1, \cdots, k$, a relation is obtained such that if the values $k=2$, $k=3$, etc. are taken in turn, the values of b_a, $a=1, 2, \cdots$, can be calculated. Thus after defining b_0 as 1, it is found that $b_1=-\frac{1}{2}$, $b_2=\frac{1}{6}$, $b_3=0$, etc. When the fractions b_{2n}, $n=1, 2, \cdots$, are expressed in their lowest terms, then the properties of the numerators and the denominators are considered mainly with the use of congruences. The methods used are quite elementary.

5. *The use of a second order correct boundary condition in the numerical solution of parabolic differential equations*, by Mr. G. W. Batten, Jr., Rice University.

The solution of the parabolic partial differential equation $\partial^2 u/\partial x^2 = F(x, t, \partial u/\partial x, \partial u/\partial t)$, subject to specified linear boundary conditions can be approximated by the solution of a difference equation subject to corresponding boundary conditions. In this paper the form of the boundary difference operator is given such that the solution of the difference equation converges to the solution of the differential equation like $O((\Delta x)^2+\Delta t)$.

6. *The discrete harmonic kernel function*, by Professor C. R. Deeter, Texas Christian University.

The discrete Laplace operator is defined by replacing the partial derivatives of the Laplacian by their corresponding difference quotients with mesh width h. For a bounded, simply connected region R, and a corresponding discrete region R_h, discrete counterparts of the Green's and Neumann functions, $G^h(z, \zeta)$ and $N^h(z, \zeta)$, are defined. The discrete harmonic kernel function of the region is defined as $K^h(z, \zeta) = N^h(z, \zeta) - G^h(z, \zeta)$. It is shown that $K^h(z, \zeta)$ is discrete harmonic, symmetric in its arguments, reproduces discrete harmonic functions on the region R_h with respect to a certain inner product, and that these properties characterize it completely.

7. *On the geometry of functions holomorphic in the unit circle, of arbitrarily slow growth, which tend to infinity on a sequence of curves approaching the circumference*, by Professor G. R. MacLane, Rice University.

It is well known that there exist functions $f(z)$, holomorphic in $|z|<1$, with $M(r)<\mu(r)$, where $\mu(r)$ is a given positive function which $\to\infty$ as $r\to 1$, and such that $\min_{|z|=r_n} |f(z)|$ approaches ∞ as $n\to\infty$. Here $r_n\to 1$ is an appropriately chosen sequence. Such functions may be constructed by the use of gap series or via an infinite product. The object of the present note is to construct such a function geometrically by starting with the Riemann surface S onto which $w=f(z)$ maps $|z|<1$.

8. *Velocities in two-dimensional potential flow*, by Professor George Copp, North Texas State College.

By using the growth of circulation about an airfoil computed by Herbert Wagner, a method is developed for computing velocities in unsteady flow in the vicinity of an airfoil in a wind tunnel. The computed velocities agree closely with measured velocities given by P. B. Walker in *Experiments on the Growth of Circulation about a Wing*.

9. *The summability of some Newton series*, by Professor Louis Brand, University of Houston.

The Newton series (in factorial powers) which are the analogues of the power series for e^x, $\cos x$, $\sin x$; e^{-x}, $\cosh x$, $\sinh x$ represent respectively the functions 2^x, $2^{\frac{1}{2}x}\cos\frac{1}{4}x$, $2^{\frac{1}{2}x}\sin\frac{1}{4}x$; 0, 2^{x-1}, 2^{x-1}. If λ and μ denote the abscissas of convergence and absolute convergence of these series, $\lambda=-1$, $\mu=0$ for the first three, $\lambda=0$, $\mu=0$ for the last three. The first three series are Cesàro-summable of order n when $-(n+1)<x\leqq -n$, $n=1, 2, \cdots$. More generally, the series $\sum_{k=0}^{\infty} x^{(k)}/k!$ is Cesàro-summable of order p (not necessarily an integer) when $x>-(p+1)$. When $x=0, 1, 2, \cdots$ the convergence of this series is of an especially simple character for it terminates after $n+1$ terms. When $x=-1, -2, -3, \cdots$ the Euler transform of this divergent series terminates after n terms

and represents 2^x. Thus the series is Euler-summable to 2^x by means of a *finite* series of n terms when $x=-n$, a negative integer.

10. *Estimation of parameters in a translated log-normal distribution with incomplete data*, by Professor P. D. Minton and Mr. Frederick Backer, Jr., Southern Methodist University.

It is assumed that survival times of patients treated for certain diseases are distributed as a translated log-normal random variable. Upon analysis of the results of the treatment, it is often found that some patients have withdrawn from treatment and others are still living when the results are being analyzed. A method originating with Lea is generalized and applied to cases where patients are still living at the time the experiment is stopped. This method gives maximum likelihood estimates for the parameters to be estimated in the assumed translated log-normal distribution.

11. *Some decomposition theorems for some classes of matrix summability operators*, by Professor E. P. Kelly, Jr., Stephen F. Austin State College.

Certain subclasses of matrix summability operators are considered as subspaces of the vector space of matrix summability operators on the set of bounded sequences of real numbers. Let T_c denote the subspace of all conservative matrix summability operators, T_b the subspace of matrix summability operators which map bounded sequences into convergence sequences, T_0 the subspace of matrix summability operators which map bounded sequences into null sequences, and $T_s \subset T_c$ the subspace of matrix summability operators which map null sequences into null sequences. A proof was given that T_c has the coset decomposition $T_c/T_0 = T_b/T_0 \oplus T_s/T_0$. Other theorems of this type are stated.

12. *Regularized set operations*, by Professor Arlen Brown, Rice University.

Define $A^{\sim} = A^{0-}$ (in a topological space) and new set operations by applying the operation "$\sim$" to the old set operations. The sets A such that $A = A^{\sim}$ ("regularly closed" sets) form a Boolean ring with respect to these operations.

13. *Holmgen-Riesz ($H=R$) transform equations of Riemannian type*, by Professor M. A. Al-Bassam, Texas Technological College.

Let α_i, a_i be numbers, $R(n-w)>0$ and $z(x) \in C^n$ on $[a, b]$, and

$$E: I^{-w} \prod_{i=1}^{3} (x-a_i)^{\alpha_i} I^{-1} (x-a_i)^{1-\alpha_i} I^{w-1} z = 0,$$

where the transform $I^{\alpha} f = D_x^n / \Gamma(\alpha+n) \int_a^x (x-t)^{\alpha+n-1} f(t)\,dt$, $R\alpha+n>0$ $(n=0, 1, \cdots)$, and $f \in C^n$ on $[a, b]$. Then it is shown that: (1) E is a differential equation of Riemannian type if and only if $F: w - \sum_{i=1}^{3} \alpha_i + I = 0$, a condition which is satisfied by the indices of Riemann P-function; otherwise E is a differential-integral equation of Riemann-Volterra type. (2) E is reduced to the Gauss's equation if for fixed i (say $i=2$) $a_2 \to \infty$, and by the operational properties of the transform the twenty-four Kummer's solutions have been obtained for this case.

14. *Ratio estimators in the balanced incomplete block design*, by Professor V. Seshadri, Southern Methodist University.

This paper proposes an estimator for the ratio of the block variance to the error variance σ_b^2/σ_e^2 in a balanced incomplete block design. This estimator is shown to be unbiased and then its variance is compared with the variance of another unbiased estimator. The difference between the variances is expressed as a quadratic function of the ratio σ_b^2/σ_e^2 and the roots of this quadratic have been examined. It is proved that the proposed estimator is uniformly better than the existing estimator for all designs that are possible.

15. *A sufficient condition that the topological space of a topological group be a Moore space*, by Mr. L. R. Carry, North Texas State College.

This paper presents a notion of point and region in a topological group; then defines a sequence of collections of regions which satisfy Professor Moore's Axiom 0 and Axiom 1, conditions 1), 2), and 3). The following theorem is then presented and proved.

THEOREM. *Let G denote a topological group. If the identity element in G admits a countable basis in the topological space G, then the topological space G is a Moore space.*

16. *An imbedding of a ring in a ring with unity*, by Professor D. E. Edmondson, University of Texas.

If R is a ring, a ring with unity is defined, R', and a homomorphism of R into R' is defined, with the properties that 1) the mapping is an isomorphism if R has a unity or no divisors of zero and 2) R' has no divisors of zero if R has no divisors of zero, and 3) concept of characteristic does not enter into the construction.

17. *Trace preserving isomorphism of operator algebras*, by Mr. Carl Pearcy, Rice University.

The author discussed the question of when a trace preserving isomorphism between two operator algebras on a finite-dimensional (real, complex) vector space is implemented by a nonsingular (orthogonal, unitary) operator.

18. *Cluster sets and pseudoanalytic functions*, by Mr. A. A. Armendarez, Rice University.

A function is said to be pseudoanalytic in a domain D if in D it is (a) an interior mapping in the sense of Stoilow, (b) it has continuous first partial derivatives, and (c) its Jacobian is positive except on at most a countable set. For a function pseudoanalytic in the unit circle the notions of cluster set $C(f, e^{i\theta})$ and of radial boundary cluster set modulo a set $E(C_{R-E}(f, e^{i\theta}))$ were discussed.

THEOREM. *Let $f(z)$ be pseudoanalytic on $|z|<1$ and let its modulus have radial limit 1 everywhere on $|z|=1$ except possibly on a set of capacity zero. If E is an arbitrary set of capacity zero on $|z|=1$, then for every $e^{i\theta}$, $f(z)$ takes on in every neighborhood of $e^{i\theta}$, $C(f, e^{i\theta})-C_{R-E}(f, e^{i\theta})$* except for at most a set of capacity zero.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 35th meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at West Virginia University on May 6, 1961. There were 118 persons present, including 63 members of the Association.

At the business meeting several items were acted upon. (1) The Committee on High School Contest Examinations indicated that approximately $750 had been accumulated over the past few years. It was voted that these funds be used to pay expenses of visiting lecturers to high schools in the area. (2) It was reported that the Association of Teachers of Collegiate Mathematics in West Virginia had passed a resolution and requested the endorsement of this resolution by the Allegheny Mountain Section. In essence, the resolution endorses the spirit of the CUPM recommendations and offers specific modifications for the state of West Virginia. The resolution contains detailed proposals concerning training of teachers and requirements for admission to West Virginia colleges, and recommends consideration of a mathematics course in the general education program. The resolution was endorsed by the Section, without dissent. (3) Officers for the next two years were elected: Chairman, Professor Evan Johnson, Jr., Pennsylvania State University; Secretary-Treasurer, Professor W. A. Beck, Chatham College; members of the Executive Committee, Dr. B. H. Mount, Westinghouse Corporation, Pittsburgh, and Professor I. D. Peters, West Virginia University.

Professor R. C. Buck, Chairman of the Committee on the Undergraduate Program in Mathematics, delivered the invited address. Professor Buck discussed the organization and the various areas of interest of the CUPM panels and summed up the major recommendations of the Committee. The floor was open for general discussion at the close of the address.

The following papers were presented:

1. *An elementary operator solution of the heat equation*, by Professor L. R. Bragg, West Virginia University.

Consider the Cauchy problem $u_t(x, t)=u_{xx}(x, t)$, $u(x, 0)=\phi(x)$. The author uses the series representation of the operator $\exp(tD^2)$ to give an elementary introduction to this problem. Through the linear and multiplicative properties of this operator, a class Ω of analytic initial functions $\phi(x)$ is built up that gives rise to closed analytic solution of the above Cauchy problem. One such elementary result is: Let $\phi(x) \in \Omega$. Then $P(x)\phi(x) \in \Omega$ for $P(x)$ a polynomial. It is also observed that this operator yields the usual results when formally applied to the Fourier series representation of a function.

2. *Operational formulas connected with generalized Hermite polynomials*, by Mr. H. W. Gould, West Virginia University.

This paper develops the two operational relations $(x+hrd^{r-1})^n f(x)=\exp(hD^r)x^n \exp(-hD^r)f(x)$ and $(D+hrx^{r-1})^n f(x)=\exp(-hx^r)D^n \exp(hx^r)f(x)$ which include well-known results involving the Hermite polynomials when $r=2$. A further generalization is indicated in terms of the Bell polynomials.

3. *A class of additive arithmetical functions*, by Mr. R. L. Duncan, Pennsylvania State University.

A discussion and proof of this result will appear in the Mathematical Notes Section of this MONTHLY.

4. *A generation of high school calculus*, by Professor Emeritus J. H. Neelley, Carnegie Institute of Technology.

This paper will appear in Mathematical Education Notes of this MONTHLY.

5. *The teaching of mathematics for management careers*, by Mr. Carlos Fallon, Radio Corporation of America, Moorestown, New Jersey.

The classical undergraduate curriculum leading to the calculus is now competing with special courses in modern mathematics for students of the biological and social sciences. Similarly, the mathematics of finance is facing rivals in the areas of decision theory and of strategy. Modern mathematics has become a most useful part of the decision-making process. It is suggested, therefore, that a program of modern mathematics be taught, not as *the* mathematics of this or that field of concentration, but *as mathematics*, having for its principal supporters, science, engineering and business administration undergraduates, but offered to all students on the campus.

6. *Assertion vs consideration in mathematical exposition*, by Professor W. A. Beck, Chatham College.

Attention is drawn to the dual role which statements play in mathematical exposition—on the one hand as merely considered, on the other as asserted. More extensive use of some sign of assertion is encouraged in symbolic logic in order to reflect this distinction in the character of statements. Illustrations are drawn from standard usage of proofs by induction, proofs by contradiction, statements of equality, and statements involving quantifiers.

7. *Mixed strategies: a geometric approach*, by Professor F. H. Steen, Allegheny College.

A geometric solution for zero-sum two person games was discussed.

EVAN JOHNSON, JR., *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The fortieth annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, Urbana, Illinois, on May 12–13, 1961. Professor Douglas Daly, Chairman of the Section, presided at all sessions. There were 84 persons in attendance, including 67 members of the Association.

The following officers were elected to serve for the coming year: Chairman, Professor T. E. Rine, Illinois State Normal University; Vice-Chairman, Professor Anice Seybold,

North Central College; Secretary-Treasurer, Professor Wayne McGaughey, Bradley University.

The Friday evening banquet speaker was Professor J. L. Doob, University of Illinois. He showed slides which were taken in Moscow and Leningrad and told of his experiences during the three weeks he spent in Russia as an exchange professor of mathematics, sponsored by the Academy of Science.

Following a brief welcome by Professor M. M. Day, Chairman of the Department of Mathematics, University of Illinois, the following program was presented.

1. *A note on Vandermonde's convolution*, by Professor Michael Skalsky, Southern Illinois University.

By use of Lagrange's formula for the inversion of power series, the following combinatorial identity was proved:

$$\sum_{k=0}^{n} A_k(a,b)\cdot A_{n-k}(c,b) = A_n(a+c,b),$$

where

$$A_n(a,b) = \binom{a+bn}{n}\frac{a}{a+bn}.$$

This identity is valid for any numbers a, b, and c. If $b=0$, it reduces to the well-known Vandermonde's convolution

$$\sum_{k=0}^{n}\binom{a}{k}\binom{c}{n-k} = \binom{a+c}{n}.$$

2. *The psychological appeal of deductive proof*, by Professor Gertrude Hendrix, University of Illinois.

The primitive fascination of power to foretell provides a strong motive for first experience with formal proof. Again and again one finds that he could have derived results obtained previously by much labor. This repeated success in prediction—this finding that things learned by experiment could have been foretold by deduction—pays off as confidence in deductive proof and taste for proving. Thus do students acquire an *inductive* basis for faith in *deduction*. In the beginning *everyone* needs to prove things he already knows.

3. *The CUPM recommendations for the training of teachers*, by Professor Rothwell Stephens, Knox College.

The underlying philosophy of the CUPM recommendations was discussed and progress made in implementing the recommendations was reported.

4. *The coconut problem*, by Mr. Clyde Bridger, Illinois Department of Public Health, Springfield.

During the day, p men and a monkey gathered T coconuts. That night each man in turn went to the pile, divided it into p equal parts, tossed the residue of k coconuts to the monkey, and hid his share. Require that $0\leqq k<p$ and let R be the remainder after the last man took his share. Then $p^p[R+k(p-1)]=(p-1)^p[T+k(p-1)]$. The general solution is $T=p^pt-k(p-1)$ and $R=(p-1)^pt-k(p-1)$, where t is a positive integer. For the classical solution, let $p=5$, $t=1$, $k=1$. Then $T=3121$ and $R=1020$.

5. *Similarity transformations on symmetric matrices*, by Professor John Christiano, Northern Illinois University.

The purpose of the paper was to demonstrate how to construct similarity transformations $R=R^{-1}$ $(RR^{-1}=1)$ for certain types of symmetric matrices so that a given matrix A when transformed into RAR^{-1} is either a diagonal matrix or one whose elements appear in a form suitable for evaluating the characteristic equation $|A-I\lambda|=0$. Symmetries and "combination" of symmetries

of four types were considered.

6. *Simple connectivity*, by Professor M. H. Heins, University of Illinois. (By invitation)

Definitions of simple-connectivity employed in the elements of the theory of functions of a complex variable were examined and related.

7. *The differential equation of a vibrating beam*, by Professor Earl McKinney, Northern Illinois University.

The vibration of a single span beam subjected to constant end and uniformly distributed axial load was considered. The equation of motion can be developed through moment analysis or, in the case of deflections about the equilibrium position, by the use of energy methods. This paper considered the determination of the equations of motion through the use of Rayleigh's principle and techniques of the variational calculus.

8. *Creativity and the search for beauty*, by Professor Rose Lariviere, University of Illinois, Navy Pier.

An awareness of the continuity of mathematical ideas contributes to the feeling of competence necessary for creative activity. Instances where continuity can be emphasized and elegance improved were given, and the retention in or restoration to the curriculum of certain topics was urged for their asethetic value regardless of their practical importance or unimportance.

9. *The fragilities of logic*, by Professor Rubin Gotesky, Department of Philosophy, Northern Illinois University, introduced by R. J. Cormier. (By invitation)

A "fragility of logic" is described by the author as consisting of two characteristics (a) a breakdown in logical use and (b) an attitude of disbelief that the "breakdown" is serious. The author discusses a number of important fragilities: (1) the impossibility-possibility fragility, (2) the disagreement fragility, (3) the familiarity fragility and (4) the rules fragility. Examples are given. The essential point of the author is that logic is a tool, an instrument. Its function and structure are determined by the specific problems, objectives, goals which arise in the life of man. There is no ultimate logic. There are only logics suited to and appropriate for given cultural and theoretical needs.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE OHIO SECTION

The forty-fifth annual meeting of the Ohio Section of the Mathematical Association of America was held at Ohio Wesleyan University, Delaware, Ohio, on Saturday, May 6, 1961. Professor Wade Ellis, Chairman of the Section, presided at the morning and afternoon sessions. There were 82 persons registered in attendance, including 71 members of the Association.

Officers selected for the coming year are: Chairman, Professor R. L. Wilson, Ohio Wesleyan University; Secretary-Treasurer, Professor Foster Brooks, Kent State University; third member of the Executive Committee, Professor Charles Saltzer, University of Cincinnati; Program Committee, Professor W. E. Restemeyer, University of Cincinnati, Chairman, Professor Clarence Heinke, Capital University, and Mr. J. W. Warner, College of Wooster.

The following papers were presented:

1. *On Boolean algebras*, by Professor Wade Ellis, Oberlin College. (Chairman's Address)

Current procedures reviewing and reconstructing mathematics curricula have apparently given inadequate consideration to the desirability of including some topics from Boolean algebras. For such algebras axiomatic bases can be brief without requiring more than elementary mathematical experience for logical development. Systems constructed on such bases have in them, at a very early stage, features which permit interesting interpretations and applications. Later developments in such systems stimulate the student's curiosity about certain modern applications of mathematics in much the same way that a study of classical geometry stimulates interest in its

classical application and interpretation.

2. *The reducibility of polynomials*, by Professor R. L. Wilson, Ohio Wesleyan University.

Let $p(x)$ be a polynomial with coefficients in the field F. This paper presents a finite method for determining whether $p(x)$ is reducible over F. If $q(x) \mid p(x)$, the product of the zeros of $q(x)$ must be in F. If $p(x)$ is of degree n and $q(x)$ of degree m, the product of the zeros of $q(x)$ is one of the C_m^n products $Q_i[i=1, \cdots, C_m^n]$, of m of the n zeros of $p(x)$. An induced polynomial is formed for which the Q_i are zeros, and the zeros of this polynomial are investigated. This investigation for all $m \leq [\frac{1}{2}n]$ provides the factors of $p(x)$ or indicates irreducibility.

3. *The Committee on the Undergraduate Program in Mathematics*, by Professor J. H. McKay, Michigan State University Oakland.

The activities, recommendations, and current areas of concern to CUPM during the past year were briefly summarized.

4. *Some conjectures associated with the Goldbach conjecture*, by Professor I. A. Barnett and Mr. Ted Cook, University of Cincinnati, presented by Professor Barnett.

The first conjecture, a stronger form of the Goldbach conjecture for odd numbers, says that every odd number $2k-1=x+2y$, where x and y are primes, $k \geqq 4$. The second conjecture is that it is possible to find a representation of every odd number of the form $6k+1$ or $6k+5$ as $2x'+3y'$ (x', y' prime) $k \geqq 3$, for which either the x' or y' appears as one of the primes in *some* representation of $2k-1$ as the sum of a prime plus the double of a prime. Both conjectures have been verified to about 15,000. If $6k+3$ is such that $2k-1$ is prime, then $6k+3$ may be written $6+3y'$ (y' prime). If, however, $2k-1$ is composite, then there is no representation of $6k+2$ as the double of a prime plus the triple of a prime.

5. *Computers and automata*, by Professor Charles Saltzer, University of Cincinnati. (By invitation).

Turing machines are defined. Some of the results of the theory concerning noncomputability and universal Turing machines are derived. Programming of Turing machines is described and the interpretation of computers and computer programs as Turing machines is given.

6. *Classroom note on an approach to Ceva's theorem*, by Professor C. H. Heinke, Capital University.

The theorem of Ceva affords an excellent opportunity, in relatively elementary mathematics, for students to experience discovery of a general proposition through a careful study of some of its special cases that are known to them. The "medians concurrent" theorem of high school geometry suggests the question; the "interior angle bisectors concurrent" theorem suggests an answer; the "altitudes concurrent" theorem suggests a proof of the general theorem. Although now well known and seemingly elementary, this theorem, requiring nothing not known to Euclid, was not published until 1678!

7. *A general theory of sums and products*, by Professor A. A. Johnson, Ohio Wesleyan University.

Category-theoretic definitions of sums and products are given in which the product is the dual of the sum. Examples from set theory include sums, (Cartesian) products, unions, and intersections. Other examples are topological sums and Cartesian products of topological spaces, logical sums and products of sentences, least upper bounds and greatest lower bounds, direct sums and Cartesian products of R-modules, tensor products, free R-modules, etc. Associativity and commutativity of such sums and products is proved. It is shown that distributivity does not always hold. Quotients and differences also arise in the theory.

8. *Some generalizations of the Fibonacci sequence*, by Professor S. E. Ganis, Ohio Wesleyan University.

The sequence $\{f_n\}$ is defined by $f_1=f_2=1$, $f_n+f_{n+1}=f_{n+2}$, $n \in J^+$. The well-known relations

(1) $f_{n-1}f_{n+1}-f_n^2=(-1)^n$ and (2) $f_{n-2}f_{n+2}-f_n^2=(-1)^{n+1}$ generalize for $s\in J^+$ as (3) $f_{n-s}f_{n+s}-f_n^2=(-1)^{n+s-1}f_s^2$. (4) $f_{n-2}f_{n+2}-f_{n-1}f_{n+1}=2(-1)^{n+1}$ generalizes as (5) $f_{n-s-1}f_{n+s+1}-f_{n-s}f_{n+s}=(-1)^{n+s}f_{2s+1}$. For $s_1,\ s_2\in J^+$ (3) and (5) further generalize as (6) $f_{n-s_1}f_{n+s_1}-f_{n-s_2}f_{n+s_2}=(-1)^{n-1}[(-1)^{s_1}f_{s_1}^2-(-1)^{s_2}f_{s_2}^2]$.

FOSTER BROOKS, *Secretary*

THE MAY MEETING OF THE OKLAHOMA SECTION

The Spring Meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma State University, Stillwater, Oklahoma, on Friday, May 12–13, 1961. Professor J. A. Nickel, Chairman of the Section, presided. There were 72 persons registered, including 57 members of the Association.

The following research papers were presented on Friday:

1. *Conformally differentiable points of arcs*, by Mr. Louis De Noya, Oklahoma State University.

2. *Nonoptical illusion*, by Mr. J. W. Schestedt, Stigler, Oklahoma.

3. *A cross and scalar product in function spaces*, by Mr. E. E. Slaughter, University of Oklahoma.

4. *A theorem related to a theorem of E. Helly*, by Mr. A. G. Haddock, Oklahoma State University.

5. *Indexed systems of neighborhoods for general topological spaces*, by Mr. Allen Davis, University of Oklahoma.

6. *Combinatorial equivalence of matrices*, (invited hour address) by Professor R. A. Good, University of Oklahoma and University of Maryland.

7. *An invariant subspace of a differential operator*, by Dr. J. E. Scroggs, University of Arkansas.

8. *Some applications of the Ramanujan function*, by Professor C. A. Nicol, University of Oklahoma.

9. *Nondegenerate convex cones*, by Mr. R. G. Laatsch, Oklahoma State University.

10. *Equivalence relations and equals relations*, by Professor W. A. Rutledge, University of Tulsa.

11. *Harmonic transformations*, by Professor N. A. Court, University of Oklahoma.

Friday night there was an evening meeting of Oklahoma college teachers to review and discuss the teacher training recommendations of the CUPM. Dr. Robert Wisner, Executive Director of CUPM, was the principal speaker.

Saturday, May 13, was also devoted to an investigation and discussion of the CUPM recommendations. Representatives of the mathematics departments of sixteen of the seventeen Oklahoma institutions of higher education that give teacher training, and representatives of several private institutions, were present and recommended that a three-step program for Oklahoma colleges in meeting CUPM suggestions for teaching training at the secondary level be adopted on a state-wide basis. This program is scheduled for completion by June, 1965. They also recommended that intermediate algebra, business mathematics, general mathematics, solid geometry, and methods of teaching mathematics be dropped for college credit by June, 1964. These courses, according to the proposed program, will not apply towards certification and accreditation of mathematics teachers after June, 1962, even if kept in the mathematics program.

R. V. ANDREE, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-ninth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin, Madison, Wisconsin, on May 13, 1961. Professor Henry Van Engen, Chairman of the Section, presided. This meeting was held jointly with the May meeting of the Wisconsin Mathematics Council and there were 129 present, including 59 members of the Association and 70 members of the Wisconsin Mathematics Council.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Earl Swokowski, Marquette University; Vice-Chairman, Professor G. L. Bullis, Wisconsin State College and Institute of Technology, Platteville; Secretary-Treasurer, Professor E. F. Wilde, Beloit College.

The following papers were presented:

Morning Session for members of the Association

1. *A problem in matrix theory*, by Professor C. B. Hanneken, Marquette University.

Let $F=GF(p)$ and $\mathfrak{M}_n$ be the ring of $n\times n$ matrices over F. A nonderogatory matrix H with irreducible characteristic equation generates a subfield $F_H=\{\sum_{i=0}^{n-1} a_iH^i \mid a_i, \text{ scalars}\}$ of $\mathfrak{M}_n$ of order p^n. There exists a nonderogatory matrix $A\in\mathfrak{M}_n$ that generates the automorphism group $\{A\}$ of F_H. Furthermore $A^n=I$ and $f(x)=x^n-1$ is the minimal polynomial of A. Let $N_A(p^n)$ be the number of distinct subfields having the same automorphism group $\{A\}$. It follows that $N_A(p^n)=\omega_n(p)/n(p-1)$, where $\omega_n(p)$ is the order of the multiplicative group of the ring $\Omega=\{\sum_{i=0}^{n-1} a_iA^i \mid a_i \text{ scalars}\}$. Clearly Ω is a commutative ring with I with nonzero divisors and is isomorphic to $F[x]/(x^n-1)$. If $\alpha=\sum_{i=0}^{n-1} a_iA^i\in\Omega$, then $|\alpha|=\prod_{j=0}^{n-1}(\sum_{i=0}^{n-1} a_i\rho^{ij})$, where ρ generates the cyclic group of roots of $f(x)$. If $n\mid p-1$, the order $\omega_n(p)$ is found to be $(p-1)^n$.

2. *Some decompositions of a Euclidean space*, by Professor L. F. McAuley, University of Wisconsin.

Suppose that $f: X\to Y$ is continuous. Given conditions on either X or f, what can be said about Y? In particular, when is Y homeomorphic to X? An approach to such problems is that of studying various subsets of X which map onto points. What upper semicontinuous decompositions of a metric space X (in particular, X is Euclidean n-space E^n) define continuous mappings onto a space homeomorphic to X? Simple examples of u.s.c. decompositions of E^2 are given which lead up to a theorem due to R. L. Moore that an upper semicontinuous decomposition of E^2 yields a space homeomorphic to E^2 if the elements of the decomposition are compact continua which do not separate E^2. Results are mentioned concerning u.s.c. decompositions of E^3 which yield spaces homeomorphic to E^3.

3. *Some aspects of general quadratic transformation*, by Professor H. P. Pettit, Marquette University.

The general quadratic transformation falls naturally into four cases: (a) double pole at $z=\infty$, (b) simple poles at $z=\infty$, $z=\xi$, (c) simple poles at $z=\xi_1$, $z=\xi_2$, (d) double pole at $z=\xi$. Analysis of the transformation of an algebraic curve of order m, with circular points of multiplicity r and the poles of multiplicity k_i is made through use of the algebraic theory of plane curves. (a) A branch with ordinary asymptote transforms into a parabolic branch. (b) Transformed curve of order $3m-2r-2k$, circular points of multiplicity $m-k$, origin of multiplicity $m-2r$. (c) Transformed curve of order $4m-2r-(k_1-k_2)$, circular points of multiplicity $2m-2r-k_1-k_2$, origin of multiplicity $2m-4r$. Case (d) was not included in this report.

Morning Session—Wisconsin Mathematics Council

4. *What will we be teaching when the shouting and tumult dies down?*, by Professor C. T. Brumfiel, University of Michigan.

Afternoon Session—joint session of the Wisconsin Section M.A.A. and the Wisconsin Mathematics Council.

5. *Crises: past and present*, by Professor R. C. Buck, University of Wisconsin.

A brief summary of the changes (and lack of changes) in mathematics education at the college level, starting at 1900, and ending with a summary of the work of the Committee on the Undergraduate Program in Mathematics through June, 1961. Emphasis was placed upon the work of the panel on Teacher Training, the panel on Mathematics for Physical and Engineering Sciences, and upon the work directed toward a general statement of the objectives of college mathematics.

6. *The status of mathematics teaching in Wisconsin secondary schools*, by Mr. A. M. Chandler, Department of Public Instruction, Madison.

Research shows that little mathematics talent can be awakened later if it has not had opportunity to be nurtured during the high school years. In Wisconsin we are beginning to see some worthwhile trends. Many schools are including more mathematics courses in their offering; revising present course offerings; developing inservice training in mathematics; increasing enrollment in mathematics classes through guidance and counseling programs; and more effectively assigning mathematical personnel. As in other states, Wisconsin is facing a shortage of well-qualified mathematics teachers. In an effort to improve the quality of mathematics teachers, the state department of public instruction has raised its certification requirements.

E. F. WILDE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, Spring, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962

KENTUCKY, University of Kentucky, Lexington, Spring, 1962.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 2, 1961.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan Ann Arbor, March 24, 1962.

MINNESOTA, Moorhead State College, November 4, 1961.

MISSOURI, Missouri School of Mines, Rolla, Spring, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14 1962.

NEW JERSEY, St. Peter's College, Jersey City, November 4, 1961.

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO

OKLAHOMA, Oklahoma City University, October 27, 1961.

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, South Dakota School of Mines and Technology, Rapid City, Spring, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

SOUTHERN CALIFORNIA, Long Beach State College, March 9, 1962.

SOUTHWESTERN

TEXAS, Rice University, Houston, April, 1962.

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, Spring, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.

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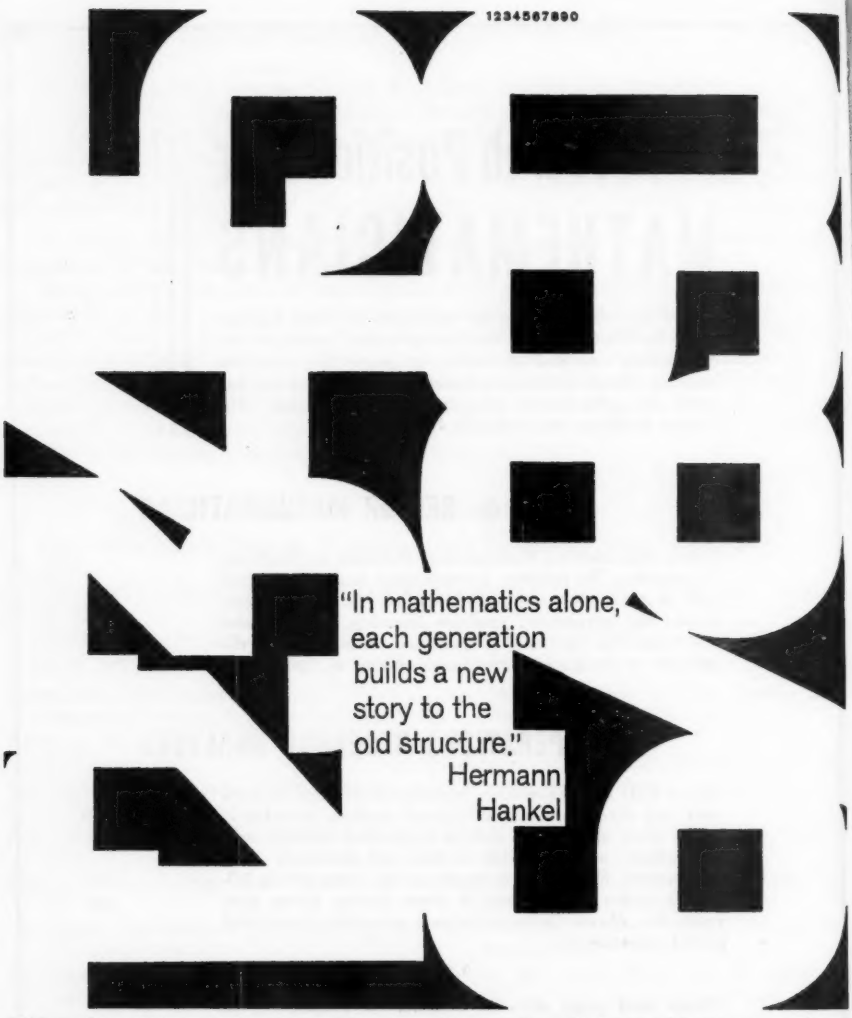

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